GAZETA DE MATEMÁTICA
N.º 50 — Dezembro de 1951

On a certain arithmetical identity related to the doubly periodic functions of the second and third kinds (*)

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1. **Introduction.** The problem of representing the doubly periodic functions of the first (i. e. elliptic), second and third kinds (using a terminology introduced by HERMITE) in various types of trigonometric series has been discussed by many authors including the distinguished Portuguese mathematician FRANCISCO GOMES TEIXEIRA. To TEIXEIRA we owe a systematic study (1) of the developments of the functions of the second kind, and the present writer is glad to acknowledge the influence of this paper of TEIXEIRA on his study of certain related problems. (2)

In what follows we shall be concerned with an arithmetical result which is implied by an identity which involves the JACOBI elliptic theta functions and certain doubly periodic functions of the second kind. The latter are meromorphic functions which satisfy periodicity conditions of the form

$$(1) \qquad \begin{cases} f(z+2\omega) = f(z) \\ f(z+2\omega') = c\, f(z)\,, \quad c \neq 1\,, \quad \Re\left(\dfrac{\omega'}{\omega}\right) > 0\,. \end{cases}$$

Included in this category are the following

$$(2) \quad f(z) \equiv \Phi_{abc}(z\,,v) \equiv \vartheta_1' \frac{\vartheta_a(z+v)}{\vartheta_b(z)\,\vartheta_c(v)}, \ (a\,,b=0\,,1\,,2\,,3)$$

for which, in a standard notation, (3) we have $\vartheta_1' = \vartheta_0\,\vartheta_2\,\vartheta_3$, $(2\,\omega\,,2\,\omega') = (\pi\,,\pi\,\tau)\,, 0 < \operatorname{amp}\tau < \pi$ and $c = \exp(-2iv)$. These functions were first discovered by JACOBI (4) in connection with his work on the dynamics of a rotating rigid body. HERMITE (1) was the first to obtain the Fourier Series developments of these functions and TEIXEIRA (2) has shown how HERMITE's results could be obtained as examples illustrating his general theory. For future reference we note that the function $\Phi_{111}(z\,,v)$ has the development

$$(3) \qquad \vartheta_1' \frac{\vartheta_1(z+v)}{\vartheta_1(z)\,\vartheta_1(v)} = \cot z + \cot v + \\ + 4\sum_{n=1}^{\infty} q^{2n}\left(\sum\nolimits_1 \sin 2\,(dz+\delta v)\right),$$

where the inner sum ranges over all the positive, integral divisors $d\,,\delta$ of n, and $\Im(z)\,,\Im(v)$ are less than $\Im(\pi\tau)$.

2. **The Analytical Identity.** The basic identity which is to be established is obtained by using a procedure which is in the spirit of the methods used by TEIXEIRA (loc. cit.) suitably modified to take care of the fact that we begin with a function which has the periodicity properties of a doubly periodic function of the *third* kind (3).

Consider the function

$$(4) \qquad f(y) \equiv \vartheta_3(x+y+z)\,\Phi_{111}(x+y\,,-y)\,,$$

where x and z are regarded for the time being as parameters and y is a complex variable. It follows from the properties of the theta functions that

$$(5) \qquad \begin{cases} f(y+\pi) = f(y)\,, \\ f(y+n\pi\tau) = \mathfrak{q}^{n^2} e^{2\,ni\,(y-z)} f(y)\,, \end{cases} \\ \mathfrak{q} = \exp(\pi i\tau)\,, \quad n = \text{integer}\,.$$

The function $f(y)$ has simple poles at $y = n\pi\tau$,

(*) Received April, 1951.

(1) F. GOMES TEIXEIRA, «Sur le développement des fonctions doublement périodiques de seconde espèce en série trigonométrique», *J. für die Reine und Ang. Mat.* **125**, (1901) p.p. 301-318.

(2) M. A. BASOCO (i) *Bull. Am. Math. Soc.* **37** (1931), p.p. 301-318, (ii) ibid, **38** (1932), p. p. 560-568, (iii) *Am. J. Math.* **54** (1932) p.p. 242-252.

(3) WHITTAKER AND WATSON, «*Modern Analysis*», Chap. XXI.

(4) JACOBI, *Werke*, Bd. 2, p. p. 291-351.

(1) HERMITE, *Oeuvres*, t. 4 p.p. 190 and 199-200.

(2) TEIXEIRA, loc. cit. p.p. 317-318.

(3) For the general theory of these functions, reference may be made to a monograph by APPELL in the *Mémorial des Sciences Mathématiques*, **36**, (1929). Also to M. A. BASOCO, *Acta Mathematica*, **57**.

and at $y = -x + n\pi\tau$, $n = 0, \pm 1, \pm 2, \cdots$, with residues $-q^{n^2} e^{-2niz} \vartheta_3(x+z)$ and $q^{n^2} e^{-2ni(x+z)} \vartheta_3(z)$ respectively.

Let C represent a contour in the y-complex plane composed of $(n+1)$ cells (of width π) above and n cells below the real axis and consider the auxiliary function

$$(6) \qquad \Phi(t) \equiv f(t) \cot(t-y),$$

which has poles at $t=y$, $t=n\pi\tau$ and $t=-x+n\pi\tau$. The residue at $t=y$ is $f(y)$. An application of the CAUCHY residue theorem leads, upon letting $n \to \infty$, to the result:

$$(7) \qquad \vartheta_3(x+y+z)\,\Phi_{111}(x+y,-y) = \\ = \vartheta_3(z)\,\chi(x+y, x+z) - \vartheta_3(x+z)\,\chi(y,z)$$

where,

$$(8) \qquad \chi(u,v) = \sum_{r=-\infty}^{\infty} q^{r^2} e^{-2irv} \cot(u - r\pi\tau),$$
$$q = exp(\pi i \tau).$$

Subject to the conditions noted below, we may express this result in the form

$$(9) \qquad \chi(u,v) = \cot u + 2\sum_{n=1}^{\infty} q^{n^2} \sin 2nv + \\ + 4\sum_{n=1}^{\infty} q^n \left(\sum \sin\{(\delta - d)u + 2dv\}\right),$$

where the inner sum refers to all the positive integral divisors d, δ of n such that $d < \delta$ and $\delta \equiv d$, (mod 2) and $\Im(u)$, $\Im(v)$ are less than $\Im(\pi\tau)$.

3. The Arithmetical Identity. If the expansions (3) and (9) are substitued in (7) and if we use the fact that $\vartheta_3(z) = \sum_{n=-\infty}^{\infty} q^{n^2} \cos 2nz$, we are led, after a fairly long calculation to the relation:

$$(10) \quad \begin{cases} 4\sum_{n=1}^{\infty} q^n \left\{\sum_{(a)} \sin 2\left[(\delta+i)x + (\delta-d+i)y + iz\right]\right\} = \\ = 4\sum_{n=1}^{\infty} q^n \left\{\sum_{(b)} \sin\left[(\Delta'+\Delta)x + (\Delta'-\Delta)y + \right.\right. \\ \left. + 2(\Delta - h)z\right] - \sum_{(b)} \sin\left[-2hx + (\Delta'-\Delta)y + \right. \\ \left.\left. + 2(\Delta-h)z\right]\right\} + \cot(x+y)\sum_{n=1}^{\infty} q^n \varepsilon(n)\left[\cos 2sz - \right. \\ \left. - \cos 2s(x+y+z)\right] - \\ - \cot y \sum_{n=1}^{\infty} q^n \varepsilon(n)\left[\cos 2s(x+z) - \right. \\ \left. - \cos 2s(x+y+z)\right] + \\ + 2\sum_{n=1}^{\infty} q^n \lambda(n)\left\{\sum_{(d)}\left[\cos 2rz \sin 2t(x+z) - \right.\right. \\ \left.\left. - \cos 2r(x+z)\sin 2tz\right]\right\}, \end{cases}$$

where the integers $i, d, \delta, h, \Delta, \Delta', t$ are subject to the relations:

(11)

(a) $n = i^2 + 2d\delta$, $i \gtreqless 0$, $\delta > 0$, $d > 0$,

(b) $n = h^2 + \Delta\Delta'$, $h \gtreqless 0$, $0 < \Delta < \Delta'$, $\Delta \equiv \Delta'$, mod 2.

(c) $n = s^2$, $s > 0$, $\varepsilon(n) = 1$ or 0 according as n is or is not a perfect square.

(d) $n = r^2 + t^2$, $r > 0$, $t > 0$, $\lambda(n) = 1$ or 0 according as n is or is not a sum of two squares.

The terms in (10) which involve the cotangent functions may be reduced by means of the formula

$$\sin ru \cot u = \sum_{k=0}^{r-1} \cos(r-2k)u.$$

The resulting terms when combined with the last series in (10) give the following simplified expression

$$(12) \qquad 4\sum_{n=1}^{\infty} q^n \varepsilon(n)\left\{\sum_{j=1}^{s-1}\left[\sin(jx+jy+sz) - \right.\right. \\ \left.\left. - \sin(sx+jy+sz)\right]\right\} + \\ + 4\sum_{n=1}^{\infty} q^n \lambda(n)\left\{\sum_{(d)} \sin(rx + (r-t)z)\right\}.$$

In this way it is seen that (10) contains only trigonometric terms of the form $\sin(\alpha x + \beta y + \gamma z)$ where α, β, γ are integers.

Using a procedure which was probably first used by LIOUVILLE in connection with his celebrated arithmetical formulae (LIOUVILLE, in this connection merely gave results; no proofs where given) and which in recent times as been fully justified by BELL [1] we obtain the following «paraphrase» of (10):

Let $F(x,y,z)$ be a single valued function defined for integral values of the arguments and subject only to the of parity conditions:

$$F(-x,-y,-z) = -F(x,y,z), \quad F(0,0,0) = 0.$$

Then identity (7) implies and is implied by the arithemetical identity:

$$(13) \qquad \sum_{(a)} F(\delta+i, \delta-d+i, i) = \\ = \sum_{(b)} \left\{ F\left(\frac{\Delta'+\Delta}{2}, \frac{\Delta'-\Delta}{2}, \Delta-h\right) - \right. \\ \left. - F\left(-h, \frac{\Delta'-\Delta}{2}, \Delta-h\right)\right\} + \varepsilon(n)\,T(n) + \lambda(n)\,L(n),$$

where,

(1) E. T. BELL, (I) *Trans. Am. Math. Soc.*, **22** (1921) p.p. 1-30 and 198-219, (II) *Coloquium Publications, Am. Math. Soc.*, **7**, 1928.

$$T(n) = \sum_{j=1}^{s-1} \left\{ F(j, j, s) - F(s, j, s) \right\}, \quad n = s^2, \quad s > 0.$$

and

$$L(n) = \sum F(r, 0, r - t), \quad n = r^2 + t^2, \quad r, t > 0$$

and (a), (b) refer to integral partitions (11), in which n is an arbitrary positive integer.

4. **Conclusion.** Somewhat simpler formulae may be obtained by restricting the parity $F(x, y, z)$ through either of the following conditions:

1) $F(-x, y, z) = F(x, y, z)$, $F(x, -y, -z) = -F(x, y, z)$, $F(x, 0, z) = 0$;

2) $F(-x, y, z) = -F(x, y, z)$, $F(x, -y, -z) = F(x, y, z)$, $F(x, 0, z) = 0$.

Corresponding to these more restrictive parity conditions we have respectively:

(14)
$$\sum_{(a)} F(\delta + i, \delta - d + i, i) = \sum_{(b)} \left\{ F\left(\frac{\Delta' + \Delta}{2}, \frac{\Delta' - \Delta}{2}, \Delta - h\right) - F\left(h, \frac{\Delta' - \Delta}{2}, \Delta - h\right) \right\} + (\varepsilon)\, T(n),$$

(15)
$$\sum_{(a)} F(\delta + i, \delta - d + i, i) = \sum_{(b)} \left\{ F\left(\frac{\Delta' + \Delta}{2}, \frac{\Delta' - \Delta}{2}, \Delta - h\right) + F\left(h, \frac{\Delta' - \Delta}{2}, \Delta - h\right) \right\} + \varepsilon(n)\, T(n).$$

These formulae are of interest in connection with certain results obtained by Uspensky in a series of memoirs entitled «Sur les Relations entre les Nombres de Classes des Formes Binaires et Positives» (1). In fact, formulas (14) and (15) were obtained by Uspensky using purely arithmetical methods. These methods are «elementary» in the sense that no analytical processes are used, but this does not necessarily mean that they are simple. By way of application of these results it may be said that Uspensky has used these formulae to enumerate the number of representations of a number as the sum of three squares, the enumerating function being a divisor function. Hence we see that these formulae are related to Gauss' classic enumeration in terms of the class number function for binary quadratic forms of negative discriminant.

It is clear that results analogous to (13), (14), (15) may be obtained by replacing equation (4) by the functions

$$f(y) = \vartheta_\alpha (x + y + z)\, \Phi_{abc}(u, v),$$

where u, v are linear functions of x and y.

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(1) J. V. Uspensky, *Bull. de l'Académie des Sciences de l'U. S. S. R* (1926) p.p. 547-566 and *Bull. Am. Math. Soc.*, **36** (1930) p.p. 743-754.