Finalmente as fórmulas (14') e (15) devem escrever-se

$$(14')\qquad (v^2-c^2)\,\mathbf{u}|\mathbf{r}-2\qquad \mathbf{v}|\mathbf{r}\,(\mathbf{u}|\mathbf{v}-c^2)=0$$

$$(15)\qquad \mathbf{u}|[(v^2-c^2)\,\mathbf{r}-2\,\mathbf{v}|\mathbf{r}\,.\,\mathbf{v}]=-2\,\mathbf{v}|\mathbf{r}\,.\,c^2.$$

Além destas gralhas é necessário ainda observar, como salientou o Prof. MIRA FERNANDES (1), que as soluções $u$, dadas pela equação (15), *não constituem um plano, como aí erradamente se disse.*

Fazendo

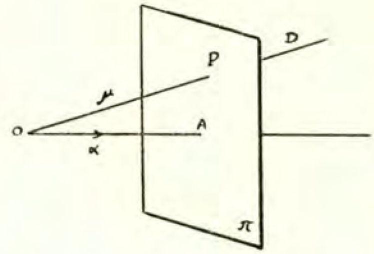
$$-\boldsymbol{\alpha}=\frac{v-c}{\mathbf{v}|\mathbf{r}}\,\mathbf{r}-2\,\mathbf{v}\,,$$

(15) toma a forma

$$\mathbf{u}|\boldsymbol{\alpha}=-2\,c^2,$$

donde se conclui que a projecção de $\mathbf{u}$ sobre $\boldsymbol{\alpha}$ é igual a $\dfrac{2\,c^2}{\mathrm{mod}\ \boldsymbol{\alpha}}$.

Sendo O uma origem qualquer a partir da qual construímos o vector $\boldsymbol{\alpha}$, e $\pi$ um plano ortogonal a $\boldsymbol{\alpha}$, cortando a direcção de $\boldsymbol{\alpha}$ no ponto $A$ tal que $\mathbf{OA}=\dfrac{2\,c^2}{\mathrm{mod}\ \boldsymbol{\alpha}}$, *em qualquer direcção* **OD** que encontra $\pi$ no ponto $P$, o vector $\mathbf{u}=P-0$ satisfaz ao problema.




As soluções de (15) são, pois, *uma* para cada direcção*:* aquela cuja projecção sobre $\boldsymbol{\alpha}$ é a constante $\dfrac{2\,c^2}{\mathrm{mod}\ \boldsymbol{\alpha}}$.

Se tivermos em conta a condição suplementar $u^2<c^2$, as soluções possíveis formarão um *cone* de raio $\mathbf{AP}=c^2\dfrac{d^2}{1+d^2}$, em que $d^2=\dfrac{(v^2-c^2)}{(\mathbf{v}|\mathbf{r})^2}\left(\dfrac{r}{2\,c}\right)$.

(1) Numa carta que nos dirigiu, donde *transcrevemos* os desenvolvimentos que seguem.

# *A classroom note on the proof of Schur's lemma* (*)

*by Hugo Ribeiro*

Let $\mathfrak{M}$ and $\mathfrak{N}$ be vector spaces over the same field, $\mathfrak{A}$ and $\mathfrak{B}$ non void families of linear mappings respectively on $\mathfrak{M}$ into $\mathfrak{M}$ and on $\mathfrak{N}$ into $\mathfrak{N}$, and $F$ a linear mapping on $\mathfrak{N}$ into $\mathfrak{M}$ such that for any $B\in\mathfrak{B}$ there is $A\in\mathfrak{A}$ for which $AF=FB$. Schur's lemma says, now, that if $\mathfrak{B}$ is irreducible (2) then either $F$ is the null mapping or it is non singular (and $\mathfrak{M}$ and $\mathfrak{N}$ have same dimension).

To prove this we first, remark that if we disregard, above, the word «linear» and substitute «vector spaces over the same field» by «sets», then we immediately obtain the following (involving just sets and mappings): if $\mathfrak{O}\subset\mathfrak{M}$ is such that for any $A\in\mathfrak{A}$, $A(\mathfrak{O})\subset\mathfrak{O}$ and if $\mathfrak{Z}$ is the set of all elements of $\mathfrak{N}$ having image in $\mathfrak{O}$, under $F$, then for any $B\in\mathfrak{B}$, $B(\mathfrak{Z})\subset\mathfrak{Z}$. (In fact, for any $B\in\mathfrak{B}$, $A\,F(\mathfrak{Z})=F\,B(\mathfrak{Z})$ for some $A\in\mathfrak{A}$, but $A(\mathfrak{O})\subset\mathfrak{O}$, hence $A\,F(\mathfrak{Z})\subset\mathfrak{O}$, so that $B(\mathfrak{Z})\subset\mathfrak{Z}$).

We have now the Schur's lemma, as a corollary, by just taking as $\mathfrak{O}$ the null subspace of $\mathfrak{M}$ and using some basic knowledge on linear mappings: for, then, $\mathfrak{Z}$ is a subspace of $\mathfrak{N}$ and, since on one hand for any $B\in\mathfrak{B}, B(\mathfrak{Z})\subset\mathfrak{Z}$ and on the other hand $\mathfrak{B}$ is supposed irreducible, we will have that either $\mathfrak{Z}$ is $\mathfrak{N}$ or it is the null subspace of $\mathfrak{N}$, hence either $F$ is the null mapping or it is non-singular (3).

(*) We are grateful to Prof. N. JACOBSON for the following references to the same type of reasoning used in this proof: N. JACOBSON, The Theory of Rings, Am. Math. Soc., 19, p. 57, R. BRAUER, On sets of matrices with coefficients in a division ring. Tran. Am. Math. Soc., vol. 49, 1941, p. 514.

(2) That is, no proper subspace of $\mathfrak{N}$ is left invariant by all $B\in\mathfrak{B}$.

(3) Presented to the 1951 meeting of the Nebraska Section of the Mathematical Association of America.