PROOF. (i) If $x \in J(x), x \neq z$, then by P2, $x^n \in J(z^n) = J(z), n = 1, 2, 3 \cdots$. According to P6 and Theorem 1 no interval of $J(z)$ contains the generator q of the subgroup Q of K. But there is an interval of $J(z)$ which contains z and x^n. Hence $x^n < q$.

(ii) Since $J(z) = z$, P4 implies that a, b is an anomalous pair if there is a u in K such that $a \in J(u)$ and $b \in J(u)$.

It may be observed that if $K = Q$, as in the example, then Theorem 3 (i) takes the following stronger form.

THEOREM 4· *Suppose* S *is a totally ordered semigroup containing zeroid elements and the kernel* K *of* S *is cyclic. Then* S *is Archimedean if and only if* J (z) = z.

PROOF. If S is Archimedean, then $J(z) = z$ according to Theorem 3. Suppose $J(z) = z$. It follows from Theorem 1 that K is an infinite subchain of S. Hence if $u \in K$, $u > z$ and m is a positive integer, there is a positive integer n such that $u^n > u^m$. Since if $a \in J(u)$, then $a^n \in J(u^n)$, it follows from P4 that $a^n > u^m$. If $u < z$, it can be shown similarly that $a^{-n} < u^{-m}$.

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Matrices whose sum is the identity matrix

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1. Let $A_i (i = 1, \cdots, m)$ be $n \times n$ complex matrices and let n_i denote the rank of A_i. In [1], p. 68 the following problem is posed:

If the matrices $A_i (i = 1, \cdots, m)$ are symmetric and $\sum_{i=1}^{m} A_i = E$ (E denotes the $n \times n$ identity matrix), show that the following statements are equivalent:

a) $A_i^2 = A_i \qquad (i = 1, \cdots, m)$

b) $\sum_{i=1}^{m} n_i = n$

c) $A_i A_j = 0 \qquad (i, j = 1, \cdots, m;\ i \neq j)$.

In [3] it is asked whether it is possible to drop the condition that the matrices $A_i (i = 1, \cdots, m)$ should be symmetric. In the present note we answer this question in the affirmative. So we no longer assume that the matrices $A_i (i = 1, \cdots, m)$ are symmetric.

We prove first that a) implies b).

If A_i is idempotent there exists a nonsingular matrix T_i such that (see [2], Vol. I, p. 226)

$$(1) \qquad A_i = T_i \operatorname{diag}(\overbrace{1, \cdots, 1}^{n_i}, \overbrace{0, \cdots, 0}^{n-n_i}) T_i^{-1}.$$

This can be proved very easily if we note that A_i is idempotent if and only if each diagonal block in its JORDAN normal form is idempotent. Let $tr\,A$ denote the trace of A. Then (1) gives

$$tr\,A_i = tr\,\mathrm{diag}(1,\cdots,1,0,\cdots,0) = n_i\,.$$

On the other hand we have $tr\,E = \sum_{i=1}^{m} tr\,A_i$ and so $n = \sum_{i=1}^{m} n_i$.

We show now that b) implies c). This has been proved by DJOKOVIČ, LANGFORD and others (see [3], where a stronger result due to R. C. THOMPSON is mentioned). For the sake of completeness we repeat a proof here.

Let $x_1^{(i)},\cdots,x_{n_i}^{(i)}$ be a basis for the range of A_i. Let x be any n dimensional vector. We have

$$x = Ex = \sum_{i=1}^{m} A_i x$$

which proves that any x can be expressed as a linear combination of the vectors $x_1^{(i)},\cdots,x_{n_i}^{(i)}$ $(i=1,\cdots,m)$. As $\sum_{i=1}^{m} n_i = n$, the number of these vectors is exactly n and so they must be linearly independent. It follows that any x can be expressed uniquely in the form $x = \sum_{i=1}^{m} x_i$ with x_i belonging to the range of A_i, namely $x_i = A_i x$. Therefore $A_j A_i x = 0$ $(i \neq j, x$ arbitrary) and so $A_j A_i = 0$ $(i \neq j)$.

Finally we show that c) implies a).

Multiplying $\sum_{i=1}^{m} A_i = E$ by A_j we get

$A_j^2 = A_j$

and the proof is complete

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Sôbre os teoremas de Zorn, de Zermelo e de Bernstein-Cantor

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Os teoremas referidos acima são deduzidos fàcilmente de um bem conhecido lema que assegura a existência de partes bem ordenadas compatíveis com uma função dada. De passagem dá-se uma demonstração simplificada dêsse lema.

1. Sejam E e F conjuntos. Uma relação *unívoca* de E para F é um subconjunto f do produto cartesiano $E \times F$ tal que se $(x,y),(x',y') \in f$ e $x = x'$, então $y = y'$. Diz-se que f é uma *função* de E para F se f é uma relação unívoca de E para F verificando a seguinte condição suplementar: para todo $x \in E$ existe pelo menos um $y \in F$ tal que $(x,y) \in f$. Se f é uma função de E para F, então para todo $x \in E$ existe um único elemento de F, indicado por