ANTOLOGIA

Natural isomorphisms in group theory (*)

by Samuel Eilenberg and Saunders MacLane
Departments of Mathematics, University of Michigan and Harvard University

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1. Introduction. — Frequently in modern mathematics there occur phenomena of «naturality»: a «natural» isomorphism between two groups or between two complexes, a «natural» homeomorphism of two spaces and the like. We here propose a precise definition of the «naturality» of such correspondences, as a basis for an appropriate general theory. In this preliminary report we restrict ourselves to the natural isomorphisms of group theory; with this limitation we can present the basic concepts of our theory without developing the axiomatic approach necessary for a general treatment applicable to various branches of mathematics.

Properties of character groups (see the definitions in § 5 below)(**) may serve to illustrate the ideas involved. Thus, it is often asserted that the character group of a finite group G is isomorphic to the group itself, but not in a «natural» way. Specifically, if G is cyclic of prime order p, there is for each generator of G an isomorphism of G to its character group, so that the proof furnishes $p-1$ such isomorphisms, no one of which is in any way distinguished from its fellows. However, the proof that that the character group of the character group of G is isomorphic to G itself is considered «natural», because it furnishes for each G a unique isomorphism, not dependent on any choice of generators.

To give these statements a clear mathematical meaning, we shall regard the character group $Ch(G)$ of G as a function of a variable group G, *together* with a prescription which assigns to any homomorphism γ of G into a second group G'.

$$\gamma : G \to G',$$

the induced homomorphism (see (5) below)

$$Ch(\gamma) : Ch(G') \to Ch(G).$$

The functions $Ch(G)$ and $Ch(\gamma)$ jointly form what we shall call a «functor»; in this case, a «contravariant» one, because the mapping $Ch(\gamma)$ works in a direction opposite to that of γ. A natural isomorphism between two functions of groups will be an isomorphism which commutes properly with the induced mappings of the functors.

With our description of a natural isomorphism, practically all the general isomorphisms obtained in group theory and its applications (homology theory, Galois theory, etc.) can be shown to be «natural». This results in added clarity in such situations. Furthermore, there are definite proofs where the naturality of an isomorphism is needed,

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(**) Ver Nota final.

especially when a passage to the limit is involved. In fact, our condition $(E\,2)$ below appears in the definition of the isomorphism of two direct or two inverse systems of groups [1].

2. *Functors.* — The definition of a functor will be given for the typical case of a functor T which depends on two groups as arguments, and is *covariant* in the first argument and *contravariant* in the second. Such a functor is determined by two functions. The *group* function determines for each pair of topological groups G and H (contained in a given legitimate set of groups) another group $T(G,H)$. The *mapping functions* determines for each pair of homomorphisms [2] $\gamma : G_1 \to G_2$ and $\eta : H_1 \to H_2$ a homomorphism $T(\gamma, \eta)$, such that

$$(1) \qquad T(\gamma,\eta):(G_1, H_2) \to T(G_2, H_1).$$

We require that $T(\gamma,\eta)$ be the identity isomorphism whenever γ and η are identities, and that, whenever the products $\gamma_2\gamma_1$ and $\eta_2\eta_1$ are defined,

$$(2) \quad T(\gamma_2\gamma_1, \eta_2\eta_1) = T(\gamma_2,\eta_1)\,T(\gamma_1,\eta_2).$$

Some functors will be defined only for special types of groups (e. g., for abelian groups) or for special types of homomorphisms (e. g., for homomorphisms «onto»).

If γ and η are both isomorphisms [3], it follows from these conditions that $T(\gamma,\eta)$ is also an isomorphism. Consequently, if the groups G_1 and G_2 and the groups H_1 and H_2 are isomorphic, the functor T gives rise to isomorphic groups $T(G_1, H_1)$ and $T(G_2, H_2)$.

3. *Examples.* — The direct product $G \times H$ of two groups may be reguarded as the group function of a functor. The corresponding mapping function specifies, for each pair of homomorphisms $\gamma : G_1 \to G_2$ and $\eta : H_1 \to H_2$, an induced homomorphism $\gamma \times \eta$, defined for every element (g_1, h_1) in $G_1 \times H_1$ as

$$[\gamma \times \eta](g_1, h_1) = (\gamma g_1, \eta h_1).$$

Then

$$(3) \qquad \gamma \times \eta : G_1 \times H_1 \to G_2 \times H_2,$$

and, whenever $\gamma_2\gamma_1$ and $\eta_2\eta_1$ are defined, one has

$$(4) \quad (\gamma_2\gamma_1)\times(\eta_2\eta_1) = (\gamma_2 \times \eta_2)(\gamma_1 \times \eta_1),$$

Except for the absence of contravariance, these conditions are parallel to (1) and (2), hence $G \times H, \gamma \times \eta$ define a functor, covariant in both G and H.

Whitney's tensor product [4] (*) $G \circ H$ of two discrete groups [5] G and H is the group function of a functor. The elements of this group are all finite sums $\Sigma\, g_1 \circ h_1$ of formal products $g_1 \circ h_1$; the group operation is the obvious addition, and the relations are $g \circ (h + h') = g \circ h + g \circ h'$, $(g + g') \circ h = g \circ h + g' \circ h$. Given two homomorphisms $\gamma : G_1 \to G_2$ and $\eta : H_1 \to H_2$, there is an induced homomorphism $\gamma \circ \eta$ of $G_1 \circ H_1$ into

(1) Pontrjagin, L., «Ueber den algebraischen Inhalt der topologische Dualitätsätze», *Mathematische Ann.*, **105**, 165-205 (1931). Lefschetz, S., «Algebraic Topology», *Am. Math. Soc. Colloquium Pub.*, **27**, 55 (1942).

(2) By a homomorphism we mean a *definite* pair of groups G_1 and G_2 and a (continuous) homomorphic mapping γ_1 of the first onto a subgroup of the second. The product $\gamma_2\gamma_1$ is defined for those pairs $\gamma_1 : G_1 \to G_2$, $\gamma_2 : G_2' \to G_3$ with $G_2 = G_2'$.

(3) By an isomorphism we mean a homomorphism of G_1 *onto* G_2 which is one-one and bicontinuous.

(4) Whitney, H., «Tensor Products of Abelian Groups», *Duke Math. Jour.*, **4**, 495-528 (1938).

(*) A notação usual é $G \otimes H$.

(5) Here and subsequently the group operation in G and in H is written as addition, whether or not the groups are abelian.

$G_2 \circ H_2$, defined for any generator $g_1 \circ h_1$ of $G_1 \circ H_1$ as

$$[\gamma \circ \eta](g_1 \circ h_1) = (\gamma g_1) \circ (\eta h_1) \in G_2 \circ H_2 .$$

Formulae (3) and (4), with the cross replaced by the circle, again hold, so that $G \circ H, \gamma \circ \eta$ determine a functor of discrete groups, covariant in both arguments.

In a similar fashion, the free product of two groups leads to a functor.

An important functor is given by the group of all homomorphisms φ of a fixed locally compact topological abelian group G into another topological abelian group H. The sum of two such homomorphisms φ_1 and φ_2 is defined for each $g \in G$ by setting $(\varphi_1 + \varphi_2)(g) = \varphi_1(g) + \varphi_2(g)$. Under this operation, all $\varphi : G \to H$ constitute a group $Hom(G, H)$: it carries an appropriate topology, the description of which we omit. For given $\gamma : G_1 \to G_2$ and $\eta : H_1 \to H_2$ and for each $\varphi \in Hom(G_2, H_1)$ we have

$$G_1 \xrightarrow{\gamma} G_2 \xrightarrow{\varphi} H_1 \xrightarrow{\eta} H_2 .$$

Consequently we define $Hom(\gamma, \eta)(\varphi) = \eta \varphi \gamma$, and verify that

$$Hom(\gamma, \eta) : Hom(G_2, H_1) \to Hom(G_1, H_2),$$

$$Hom(\gamma_2 \gamma_1, \eta_2 \eta_1) = Hom(\gamma_1, \eta_2) \, Hom(\gamma_2, \eta_1) .$$

Clearly when γ and η are identity mappings of G and H the induced mapping $Hom(\gamma, \eta)$ is the identity mapping of $Hom(G, H)$ on itself. Hence the functions $Hom(G, H)$ and $Hom(\gamma, \eta)$ determine for abelian groups a functor Hom, covariant in H and contravariant in G.

The special case when H is the group P of reals modulo 1 furnishes the character group,

$$Ch(G) = Hom(G, P), \quad Ch(\gamma) = Hom(\gamma, e)$$

where e is the identity mapping of P on itself. Therefore the character group is a contravariant functor, defined for abelian groups. Explicitly, if we express the result $\chi(g)$ of applying the character χ to the element $g \in G$ as the value (a real number modulo 1) of the bilinear form (g, χ), the definition of $Ch(\gamma)$ can be written as

$$(5) \qquad (g, Ch(\gamma)\chi') = (\gamma g, \chi'), \ g \in G, \ \chi' \in Ch(G')$$

4. Equivalence of Functors. — Let T and S be two functors which are, say, both covariant in the variable G and contravariant in H. Suppose that for each pair of groups G and H we are given a homomorphism

$$\tau(G, H) : T(G, H) \to S(G, H) .$$

We say that τ establishes a natural *equivalence* of the functor T to the functor S and that T is naturally *equivalent* to S (in symbols, $\tau : T \longleftrightarrow S$) whenever

(E 1) Each $\tau(G, H)$ is a bicontinuous isomorphism of $T(G, H)$ onto $S(G, H)$;

(E 2) For each $\gamma : G_1 \to G_2$ and $\eta : H_1 \to H_2$, $\tau(G_2, H_1) T(\gamma, \eta) = S(\gamma, \eta) \tau(G_1, H_2)$.

The first requirement insures the term-by-term isomorphism of the two group functions $T(G . H)$ and $S(G, H)$, while the second requirement is precisely the «naturality» condition. It can be shown that the condition (E 2) is implied by two special cases; the case when η is an identity, and the case when γ is an identity.

This relation of natural equivalence between functors is reflexive, symmetric and transitive. In many cases we dispense with condition (E 1), and obtain a more general concept of a «transformation» of a functor T into a functor S.

5. *Examples of Natural Equivalence.* — The well known isomorphism

$$G \cong Ch(Ch(G)) \tag{6}$$

for locally compact abelian groups, can be regarded as an equivalence of functors, and is in this sense *natural*. The right-hand side of (6) suggests the covariant functor, Ch^2, defined by iteration of the functor Ch, as

$$Ch^2(G) = Ch(Ch(G)), \quad Ch^2(\gamma) = Ch(Ch(\gamma)).$$

The left-hand side of (6) suggests the identity functor, I,

$$I(G) = G, \quad I(\gamma) = \gamma.$$

The bilinear form $(g, \chi) = \chi(g)$ determines to each character $\chi \in Ch(G)$ and each $g \in G$ a real number modulo 1; similarly the form $(\chi, h) = h(\chi)$ is defined for each $h \in Ch^2(G)$. The form (g, χ), regarded as a function of χ for fixed g, is a character h in $Ch^2(G)$ which we call $[\tau(G)]g$. Explicitly, this definition of τ reads

$$(\chi, \tau(G)g) = (g, \chi), \quad g \in G, \quad \chi \in Ch(G).$$

The validity of condition $(E\,1)$ for $\tau(G)$ is the basic theorem of character theory. The condition $(E\,2)$ asserts that in the diagram

$$\begin{tikzcd} G \arrow[r, "\tau(G)"] \arrow[d, "\gamma"'] & Ch^2(G) \arrow[d, "Ch^2(\gamma)"] \\ G' \arrow[r, "\tau(G')"] & Ch^2(G') \end{tikzcd}$$

the two paths leading from G to $Ch^2(G')$ have the same effect, or that, for each $g \in G$, both elements $\tau(G')\gamma g$ and $Ch^2(\gamma)\tau(G)g$ are identical as elements of $Ch^2(G')$. This means that, for each $\chi' \in Ch(G')$, one should have

$$(\chi', \tau(G')\gamma g) = (\chi', Ch^2(\gamma)\tau(G)g).$$

By the definition of τ, the expression on the left is simply $(\gamma g, \chi')$. By successive application to the expression on the right of the definitions of Ch, τ and Ch, we obtain

$$\begin{aligned}(\chi', Ch^2(\gamma)\tau(G)g) &= (Ch(\gamma)\chi', \tau(G)g) = \\ &= (g, Ch(\gamma)\chi') = (\gamma g, \chi').\end{aligned}$$

The identity of these results shows that we do have a natural equivalence

$$\tau(G) : G \longleftrightarrow Ch^2(G).$$

When G is finite, the isomorphism $G \to Ch(G)$ cannot be «natural» according to our definitions, for the simple reason that the functor I on the left is covariant, while the functor Ch on the right is contravariant.

As other examples of equivalences between functors, we may cite the usual isomorphisms which give the associative and commutative laws for the direct product, the tensor product and the free product. Various distributive laws, such as

$$(G_1 \times G_2) \circ H \cong (G_1 \circ H) \times (G_2 \circ H),$$

$$\begin{aligned} Hom(G_1 \times G_2, H) \cong Hom(G_1, H) \times \\ \times Hom(G_2, H),\end{aligned}$$

when established with the obvious isomorphisms, are in fact equivalences between fnnctors.

A less obvious relation between the tensor product and the functor «*Hom*» is [1]

[1] This isomorphism was established by the authors; cf. *Ann. Math.*, **44** (1943).

$$(7)\quad Hom(G, Hom(H, K)) \cong Hom(G \circ H, K),$$

where G and H are discrete abelian groups, K a topological abelian group. This isomorphism is obtained by a correspondence $\tau(G, H, K)$ which specifies for each element $\varphi \in Hom(G, Hom(H, K))$ a corresponding homomorphism in $Hom(G \circ H, K)$, defined for any generator $g \circ h$ of $G \circ H$ as

$$[\tau(G, H, K)]\ (\varphi)(g \circ h) = [\varphi(g)](h) \quad \text{in} \quad K.$$

One may show that τ does give an isomorphism, bicontinuous in the appropriate topologies. Both sides of (7) may be treated as the group functions of functors which are obtained by composition from «*Hom*» and «$\circ$». The corresponding mapping functions, for given homomorphisms

$$\gamma : G_1 \to G_2, \quad \eta : H_1 \to H_2, \quad \varkappa : K_1 \to K_2,$$

are defined by a parallel composition as

$$Hom(\gamma, Hom(\eta, \varkappa)), \quad Hom(\gamma \circ \eta, \varkappa).$$

Both functors are contravariant in G and H, covariant in K.

The naturality condition for the isomorphism τ reads

$$\tau(G_1, H_1, K_2)\, Hom(\gamma, Hom(\eta, \varkappa)) =$$
$$= Hom(\gamma \circ \eta, \varkappa)\, \tau(G_2, H_2, K_1).$$

Both sides, when applied to an element $\varphi \in Hom(G_2, Hom(H_2, K_1))$ yield a homomorphism in $Hom(G_1 \circ H_1, K_2)$. If each of these homomorphisms is applied to a typical generator $g_1 \circ h_1$ of the tensor product $G_1 \circ H_1$, straightforward application of the relevant definitions shows that the same element of K_2 is obtained in both cases; namely, $\varkappa\{[\varphi(\gamma(g_1))](\eta(h_1))\}$. One may also see directly that this expression represents the only way of constructing an element of K_2 from the elements g_1 and h_1 and the mappings $\varkappa$, φ, γ and η.

The natural isomorphism (7) has some interesting consequences. If K is taken to be the group P of real numbers modulo 1, $Hom(H, K)$ becomes the character group $Ch(H)$, and the formula may be written as

$$Hom(G, Ch\, H) \cong Ch(G \circ H).$$

Applying the functor Ch to both sides and using the natural equivalence of Ch^2 and I, we obtain the equivalence

$$G \circ H \cong Ch\, Hom(G, Ch\, H).$$

Since this is «natural», this could be used as a definition of the tensor product $G \circ H$.

6. *Generalisations* — With the appropriate definition of a normal subfunctor S of a functor T one can construct a quotient functor T/S, whose group function has as its values quotient groups (i. e., factor groups). With this operation, all the standard constructions on groups may be represented as group functions of suitable functors.

An inspection of the concept of a functor and of a natural equivalence shows that they may be applied not only to groups with their homomorphisms, but also to topological spaces with their continuous mappings, to simplicial complexes with their simplicial transformations, and to Banach spaces with their linear transformations. These and similar applications can all be embodied in a suitable axiomatic theory. The resulting much wider concept of naturality, as an equivalence

between functors, will be studied in a subsequent paper.

NOTA: A Nota que aqui incluimos destina-se a facilitar aos leitores não familiarizados com a noção de grupo dos caracteres de um grupo, a compreensão das ideias expressas pelos autores no § 1. Procuràmos respeitar o espírito do texto.

Seja V um espaço vectorial real e V^* o seu dual (i. e. o conjunto das aplicações lineares de V em $\mathbf{R}$ munido da estrutura de espaço vectorial real usual). É bem sabido que se V tem dimensão finita n, a dimensão de V^* é também n e, portanto, V e V^* são isomorfos. Mais precisamente, para cada par de bases $(e_i)_{1 \leq i \leq n}$ e $(e_i^*)_{1 \leq i \leq n}$ de V e V^*, respectivamente, existe um (e um só) isomorfismo $f: V \to V^*$ tal que $f(e_i) = e_i^*$, $1 \leq i \leq n$. Nestas condições, se $n \neq 0$, existe uma infinidade de isomorfismos de V sobre V^* nenhum dos quais se pode òbviamente considerar privilegiado em relação aos outros. Todavia, a aplicação $\Psi_V: V \to V^{**}$ que associa a cada $x \in V$ a forma linear sobre V^*, $\Psi\ (x)$ tal que $(\Psi_V(x))(f) = f(x)$ para todo o $f \in V^*$, é um isomorfismo de V sobre V^{**} considerado «natural» por não depender de quaisquer bases prèviamente escolhidaspara V e V^{**}.

Para dar um sentido preciso ao que precede, convém considerar a passagem ao dual como um *par* de funções: a primeira função associa a cada espaço V o seu dual V^* e a segunda função associa a cada aplicação linear $\gamma: V \to V'$ a transposta $\gamma^*: V'^* \to V^*$ (i. e. a aplicação linear que faz corresponder a cada $f \in V'^*$, $\gamma^*(f) \in V^*$ tal que, para todo o $x \in V$, $(\gamma^*(f))(x) = f(\gamma(x))$). Este par de funções é um *funtor*; mais precisamente, um functor *contravariante* visto que a aplicação correspondente a uma aplicação linear de V em V' é uma aplicação linear de V'^* em V^*.

Um *isomorfismo* ou *equivalência natural* entre dois fuctores S e T é uma família de isomorfismos $S(V) \to T(V)$ tal que, para cada $\gamma: V \to V'$, $S(V) \to T(V)$, $S(V') \to T(V')$ constituem com $S(\gamma)$, $T(\gamma)$ um diagrama comutativo.

Assim, se S e T são funtores *còvariantes*, a aplicação composta de $S(V') \to T(V')$ com $S(\gamma)$ é igual à composta de $T(\gamma)$ com $S(V) \to T(V)$. A situação que consideramos no início da Nota ilustra perfeitamente este conceito: se S designa o functor idêntico (que associa a cada espaço vectorial real de dimensão finita V o próprio V e a cada aplicação linear γ a própria aplicação γ) e T designa o funtor que associa a V o seu bidual V^{**} e a cada aplicação linear γ, γ^{**}, reconhece-se imediatamente que a família de isomorfismos Ψ_V é uma equivalência natural entre S e T; todavia, os funtores idêntico e «passagem ao dual» não podem ser naturalmente equivalentes em virtude de o primeiro ser còvariante e o segundo contravariante.

A. V. Ferreira

«A matemática foi criada pelos homens para satisfação das suas necessidades, e tem sido para eles, de facto, um precioso instrumento; o professor de matemática deve permanecer por isso um professor de acção...».

H. Lebesgue

«A produção industrial pode estar sensìvelmente em atraso sobre as descobertas científicas» ..... «a aplicação rápida das descobertas científicas na economia nacional supõe resolvidos um certo número de problemas económicos, institucionais, etc.».

M. Lavrentiev

PRÉLUDE (1)

Au commencement tout était morne et informe.
Le Géomètre dit: Que la lumière soit!
Et les structures là par espèces perçoit;
Du chaos émergent attraits, contours et normes.

Dès lors il structure les objets et les flèches,
Puis structure les structures, tâche sans fin.
L'application, outil premier pour Dedekind,
Se mue en morphisme, d'apparence une flèche.

Catégories en expansion dans l'Univers,
Leurs trios s'accordent, les quatuors résonnent,
Comme les quintettes, niant deux fois Zénon,

Se composent en long, et en large, un sens clair
Aux métamorphoses bien naturelles donnent.
Tout cela, dit le Sphinx (*), *pour amuser* Ζεῦν.

(*) Goethe, Faust II, Nuit classique de Walpurgis.

(1) Reprodução amàvelmente autorizada pelo Editor, da pág. V da obra «Catégories et Structures» de Ch. Ehresmann; Dunod, ed. 1965.