Expansion theorems for generalized hypergeometric functions I

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1. **Introduction**. In previous papers [(1)] and [(2)] expansions for FOX's H-function involving JACOBI polynomials and LAGUERRE polynomials respectively were established. In this paper, we generalized the low order expansions for the H-functions [(1), (2)] by LAPLACE transform techniques and employ them to obtain expansions for MEIJER's G-function and MACROBERT's E-function. The expansions given here are of general character and numerous special cases of the general theorems developed in this paper are scattered throughout the literature. Some well known results recently given by MEIJER, BAJPAI, WIMP and LUKE are shown as particular cases.

The subject of expansion formulae for generalized hypergeometric functions occupies a prominent place in the literature of special functions. Certain expansions and series of hypergeometric functions, play an important role in the development of the theory of special functions.

The expansion theorems for hypergeometric functions were given from time to time by various mathematicians, with certain restrictions in the parameters. An adequate list of references would be quite lengthy. However, the references given here together with the sources indicated in these references provide a good converge of this subject.

We now, mention in brief some interesting work on this subject. In a series of papers, MEIJER [(31) to (33)] gave expansions of a G-function in a series of related G-functions multiplied by hypergeometric polynomials ${}_{p+1}F_q(-n, a_p; b_q; z)$. LUKE, WIMP, FIELDS and other members of staff of Midwest Research Institute contributed valuable papers [(20), (21), (29), (30), (41), (42)] to the subject. Recently LAWRYNOWICZ [(26 to (28)] has given some interesting expansions, which include those of MEIJER. Most recently the author in a series of papers [(1) to (15)] has obtained a number of expansion formulae for MACROBERT's E-function, MEIJER's G-function and FOX's H-function, which on generalization will give more results very soon.

The H-function introduced by FOX [(24), p. 408], will be represented and defined as follows:

$$(1.1)\qquad \mathrm{H}_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1, e_1), \cdots, (a_p, e_p)\\ (b_1, f_1), \cdots, (b_q, f_q)\end{matrix}\right]=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js)\,z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=n+1}^{p}\Gamma(a_j-e_js)}\,ds,$$

where L is a suitable contour.

In what follows for sake of brevity (a_p, e_p) is to be interpreted as $(a_1, e_1), \cdots, (a_p, e_p)$, $(a_p)_\mu$ denotes $\prod_{j=1}^{p} (a_j)_\mu$ and the symbol $\Delta(h, a)$ represents the set of parameters

$$\frac{a}{h}, \frac{a+1}{h}, \cdots, \frac{a+h-1}{h},$$

where h is a positive integer. Also for ease in writing, we denote

$$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j = A, \quad \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j = B,$$

and $\mathrm{Re}(\delta + b_m/f_m)$ represents $\mathrm{Re}(\delta + b_j/f_j)$ where $j = 1, 2, \cdots, m$.

The following formulae are required in the proof:

If h is a positive number, $A \leqq 0$, $B > 0$, $|\arg z| < \frac{1}{2} B\pi$, $\mathrm{Re}\,\alpha > -1$, $\mathrm{Re}\,\beta > -1$, $\mu \geqq 0$, $\mathrm{Re}(\mu + \alpha + h\, b_m/f_m) > -1$, then

$$(1.2) \qquad \omega^\mu H_{p,q}^{m,n}\left[z\,\omega^h \,\middle|\, \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] = \sum_{N=0}^{\infty} \frac{(-1)^N (\alpha+\beta+2N+1)(\alpha+\beta+N+1)}{\Gamma(\alpha+1)\, N!}$$

$$\cdot H_{p+2,q+2}^{m,n+2}\left[z \,\middle|\, \begin{matrix} (-\mu, h), (-\mu-\alpha, h), (a_p, e_p) \\ (b_q, f_q), (N-\mu, h), (-1-\alpha-\beta-\mu-N, h) \end{matrix} \right] \times {}_2F_1\left[\begin{matrix} -N, N+\alpha+\beta+1; \omega \\ \alpha+1 \end{matrix} \right],$$

which follows from [(1), (3. 1)] and [(35), p. 254, (1)].

If h is a positive number, $A \leqq 0$, $B > 0$, $|\arg z| < \frac{1}{2} \beta\pi$, $\mathrm{Re}\,\alpha > -1$, $\mu \geqq 0$, $\mathrm{Re}(\mu + \alpha + h\, b_m/f_m) > -1$, then

$$(1.3) \qquad \omega^\mu H_{p,q}^{m,n}\left[z\,\omega^h \,\middle|\, \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] = \sum_{N=0}^{\infty} \frac{(-1)^N}{N!\,\Gamma(\alpha+1)}$$

$$\cdot H_{p+2,q+1}^{m,n+2}\left[z \,\middle|\, \begin{matrix} (-\mu-\alpha, h), (-\mu, h), (a_p, e_p) \\ (b_p, f_q), (N-\mu, h) \end{matrix} \right] \times {}_1F_1\left[\begin{matrix} -N; \omega \\ \alpha+1 \end{matrix} \right],$$

which follows from [(2), (3. 1)] and [(35), p. 200, (1)].

If h is a positive number, $A \leqq 0$, $B > 0$, $|\arg z| < \frac{1}{2} B\pi$, $\mathrm{Re}(h\, b_m/f_m - \rho) > -1$, then

$$(1.4) \qquad \int_0^\infty -x^{-\rho} e^{-\lambda x} H_{p,q}^{m,n}\left[z\, x^h \,\middle|\, \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx = \lambda^{\rho-1} H_{p+1,q}^{m,n+1}\left[z\,\lambda^{-h} \,\middle|\, \begin{matrix} (\rho, h), (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right],$$

which can be established by expressing the H-function in the integrand as a Mellin-Barnes type integral (1. 1), inter-changing the order of integrations and using [(18), p. 137, (1)].

The LAPLACE transform [(18), p. 219, (17)] given by

$$\int_0^\infty e^{-\rho t}\, t^{\sigma-1}\, {}_pF_q\left[\begin{matrix} a_p\,;\lambda t \\ b_q \end{matrix}\right] dt = \Gamma(\sigma)\,\rho^{-\sigma}\, {}_{p+1}F_q\left[\begin{matrix} a_p\,,\sigma\,;\lambda/\rho \\ b_q \end{matrix}\right], \tag{1.5}$$

where $p \leqq q$, $\operatorname{Re}\sigma > 0$.

2. THEOREM I.

(i) *Let none of the following quantities be negative integers:*

$$b_m/f_m + \frac{\mu - h + 1}{h}\,;\ b_m/f_m + \frac{\mu + \alpha_t - h}{h}\,;\ \nu\,;\ \beta_u - 1\,;\ b_m/f_m - a_n/e_n\,,$$

where h *is a positive number.*

(ii) *Let* $A + h(r-s) \leqq 0$, $B + h(r-s) > 0$, $|\arg z| < \frac{\pi}{2}\{B + h(r-s)\}$, $Re(\mu + \alpha_t + h\, b_m/f_m) > 0$, $Re(\nu - \alpha_t) > 0$, $Re\,\alpha_t > 0$, $Re(h\, b_m/f_m - c_r) > -1$, $Re(h\, b_m/f_m - d_s) > -1$, $Re(1 - c_r - \mu) > 0$, $Re(1 - d_s - \mu) > 0$.

(iii) *Let* p, q, r, s, t, u, m *and* n *be positive integers or zero;*

$p + r \leqq q + s - 1$ *or* $p + r = q + s$ *and* $|z\,\omega^h| < 1$,

$p + t \leqq q + (u+1) - 1$ *or* $p + t = q + (u+1)$ *and* $|z| > 1$,

$0 \leqq m \leqq q$; $0 \leqq n \leqq p$; $q + s \geqq 1$.

(iv) *Let* $r + u + 1 = s + t$.

(v) *Let* $0 < \omega < 1$, $z \neq 0$.

(vi) *Let* $\Sigma d_s - \Sigma c_r + \Sigma \beta_u - \Sigma \alpha_t - 2h\, b_m/f_m < (s-r)(1-\mu) + 2\mu + \frac{1}{2}$,

$$1 + b_m/f_m - \frac{c_r + h - 1}{h} > 0, \qquad 1 + b_m/f_m - \frac{h - \mu - \beta_u}{h} > 0.$$

Then

$$\omega^\mu H^{m,\,n+r}_{p+r,\,q+s}\left[z\,\omega^h \left| \begin{matrix} (c_r, h), (a_p, e_p) \\ (b_q, f_q), (d_s, h) \end{matrix}\right.\right] = \frac{\Gamma(1-c_r-\mu)\,\Gamma(\beta_u)}{\Gamma(1-d_s-\mu)\,\Gamma(\alpha_t)} \sum_{N=0}^{\infty} \frac{(-1)^N(\nu+2N)\,\Gamma(\nu+N)}{N!}$$
$$\times H^{m,\,n+t+1}_{p+t+1,\,q+u+2}\left[z \left| \begin{matrix} (-\mu, h), (1-\alpha_t-\mu, h), (a_p, e_p) \\ (b_q, f_q), (N-\mu, h), (-\mu-\nu-N, h), (1-\beta_u-\mu, h) \end{matrix}\right.\right]$$
$$\times\, {}_{r+u+2}F_{s+t}\left[\begin{matrix} -N, \nu+N, 1-c_r-\mu, \beta_u\,;\omega \\ \alpha_t, 1-d_s-\mu \end{matrix}\right]. \tag{2.1}$$

PROOF. We first prove (2. 1) for the case $u=0$, $t=1$ and $\alpha_1=\alpha$, that is

$$(2.2)\quad \omega^{\mu}\,\mathrm{H}^{m,n+r}_{p+r,q+s}\left[z\,\omega^{h}\left|\begin{matrix}(c_r,h),(a_p,e_p)\\(b_q,f_q),(d_s,h)\end{matrix}\right.\right]=\frac{\Gamma(1-c_r-\mu)}{\Gamma(1-d_s-\mu)\Gamma(\alpha)}\sum_{N=0}^{\infty}\frac{(-1)^N(\nu+2N)\Gamma(\nu+N)}{N!}$$

$$\times\,\mathrm{H}^{m,n+2}_{p+2,q+2}\left[z\left|\begin{matrix}(-\mu,h),(1-\alpha-\mu,h),(a_p,e_p)\\(b_q,f_q),(N-\mu,h),(-\mu-\nu-N,h)\end{matrix}\right.\right]$$

$$\times\,{}_{r+2}F_{s+1}\left[\begin{matrix}-N,\nu+N,1-c_r-\mu;\ \omega\\\alpha,1-d_s-\mu\end{matrix}\right].$$

Our proof for (2. 2) is based on induction on the parameters r and s (Note that the case $r=s=0$ is the result (1. 2) if we replace α by $\alpha+1$ and put $\nu=\alpha+\beta+1$). Multiplying both sides of (2. 2) by $\omega^{-\sigma-\mu}e^{-\lambda\omega}$, integrating with respect to ω from 0 to ∞, we get

$$\int_0^{\infty}\omega^{-\sigma}e^{-\lambda\omega}\,H^{m,n+r}_{p+r,q+s}\left[z\,\omega^{h}\left|\begin{matrix}(c_r,h),(a_p,e_p)\\(b_q,f_q),(d_s,h)\end{matrix}\right.\right]d\omega$$

$$=\frac{\Gamma(1-c_r-\mu)}{\Gamma(1-d_s-\mu)\Gamma(\alpha)}\sum_{N=0}^{\infty}\frac{(-1)^N(\nu+2N)\Gamma(\nu+N)}{N!}$$

$$\times\,\mathrm{H}^{m,n+2}_{p+2,q+2}\left[z\left|\begin{matrix}(-\mu,h),(1-\alpha-\mu,h),(a_p,e_p)\\(b_q,f_q),(N-\mu,h),(-\mu-\nu-N,h)\end{matrix}\right.\right]$$

$$\times\int_0^{\infty}\omega^{-\sigma-\mu}e^{-\lambda\omega}\,{}_{r+2}F_{s+1}\left[\begin{matrix}-N,\nu+N,1-c_r-\mu;\ \omega\\\alpha,1-d_s-\mu\end{matrix}\right]d\omega.$$

Now with the help of (1. 4) and (1. 5), we obtain

$$\lambda^{-\mu}\,\mathrm{H}^{m,n+r+1}_{p+r+1,q+s}\left[z\,\lambda^{-h}\left|\begin{matrix}(\sigma,h),(c_r,h),(a_p,e_p)\\(b_q,f_q),(d_s,h)\end{matrix}\right.\right]$$

$$=\frac{\Gamma(1-c_r-\mu)\Gamma(1-\sigma-\mu)}{\Gamma(1-d_s-\mu)\Gamma(\alpha)}\sum_{N=0}^{\infty}\frac{(-1)^N(\nu+2N)\Gamma(\nu+N)}{N!}$$

$$\times\,\mathrm{H}^{m,n+2}_{p+2,q+2}\left[z\left|\begin{matrix}(-\mu,h),(1-\alpha-\mu,h),(a_p,e_p)\\(b_q,f_q),(N-\mu,h),(-\mu-\nu-N,h)\end{matrix}\right.\right]$$

$$\times\,{}_{r+3}F_{s+1}\left[\begin{matrix}-N,\nu+N,1-c_r-\mu,1-\sigma-\mu;\ \dfrac{1}{\lambda}\\\alpha,1-d_s-\mu\end{matrix}\right].$$

Now replacing $1/\lambda$ by ω and σ by c_{r+1} and the induction on r is completed. To perform the induction on s, multiply both sides of (2. 2) by $\omega^{1-\mu-\sigma}$, replacing ω by $1/\lambda$, take the inverse LAPLACE transforms of both sides with the help of (1. 4) and [(18), p. 297, (1)] and then indentify σ with d_{s+1}.

Now we establish a relation from (2. 2) which will be finally used to obtain (2. 1).

Putting $m=1$, $n=p$, $b_1=0$, $e_j=f_j=1$ $(j=1,2,\cdots,p;\ i=1,2,\cdots,q), h=1$, replacing a_p by $1-a_p$, q by $q+1$ and b_{j+1} by $1-b_j(j=1,2,\cdots,q)$, c_r by $1-c_r$, d_s by $1-d_s$ and using the formula

$$\text{(2. 3)} \qquad H^{1,p}_{p,q}\left[z\left|\begin{matrix}(a_p,1)\\(b_q,1)\end{matrix}\right.\right]=\frac{\prod\limits_{j=1}^{p}\Gamma(1+b_1-a_j)\,z^{b_1}}{\prod\limits_{i=1}^{q}\Gamma(1+b_1-b_j)}$$

$$\times\,{}_pF_{q-1}\left[\begin{matrix}1+b_1-a_1,\cdots,1+b_1-a_p\,;\ -z\\1+b_1-b_2,\cdots,1+b_1-b_q\end{matrix}\right],\ p\leqq q,$$

and simplifying, we get

$$\text{(2. 4)} \qquad \omega^{\mu}\,{}_{p+r}F_{q+s}\left[\begin{matrix}a_p,c_r\,;z\,\omega\\b_q,d_s\end{matrix}\right]=\frac{(c_r)_{-\mu}(\alpha)_{\mu}}{(d_s)_{-\mu}}\sum_{N=0}^{\infty}\frac{(-1)^N(\nu+2N)}{N!\,(N+\nu)_{\mu+1}}$$

$$\cdot\,{}_{p+2}F_{q+2}\left[\begin{matrix}a_p,1+\mu,\alpha+\mu & ;z\\b_q,1+\mu-N,1+\mu+\nu+N\end{matrix}\right]\times{}_{r+2}F_{s+1}\left[\begin{matrix}-N,N+\nu,c_r-\mu & ;\omega\\\alpha,d_s-\mu\end{matrix}\right].$$

In (2. 4), putting $z=0$ replacing r by $r+u$, let $c_{r+\lambda}=\beta_\lambda+\mu$, $\lambda=1,2,\cdots,u$ and replace c_λ by $c_\lambda+y\delta$, $\lambda=1,2,\cdots,r$. Replacing s by $s+t$, let $d_{s+\lambda}=\alpha_\lambda+\mu$, $\lambda=1,2,\cdots,t$ and replace d_λ by $d_\lambda+y\delta$, $\lambda=1,2,\cdots,s$ and ω by $\alpha\omega$, we have

$$(\alpha\,\omega)^{\mu}=\frac{(c_r+y\,\delta)_{-\mu}(\beta_u+\mu)_{-\mu}(\alpha)_{\mu}}{(d_s+y\,\delta)_{-\mu}(\alpha_t+\mu)_{-\mu}}\times\sum_{N=0}^{\infty}\frac{(2N+\nu)(-\mu)_N}{(N+\nu)_{\mu+1}}$$

$$\cdot\,{}_{r+u+2}F_{s+t+1}\left[\begin{matrix}-N,N+\nu,c_r+y\,\delta-\mu,\beta_u\,;\alpha\,\omega\\\alpha,d_s+y\,\delta-\mu,\alpha_t\end{matrix}\right].$$

In the above expression let $\alpha\to\infty$, we get

$$\omega^{\mu}=\frac{(c_r+y\,\delta)_{-\mu}(\beta_u+\mu)_{-\mu}}{(d_s+y\,\delta)_{-\mu}(\alpha_t+\mu)_{-\mu}}\times\sum_{N=0}^{\infty}\frac{(2N+\nu)(-\mu)_N}{N!\,(N+\nu)_{\mu+1}}$$

$$\cdot\,{}_{r+u+2}F_{s+t}\left[\begin{matrix}-N,N+\nu,c_r+y\,\delta-\mu,\beta_u\,;\omega\\d_s+y\,\delta-\mu,\alpha_t\end{matrix}\right].$$

Now we replace μ by $\mu + y\delta$, c_r by $1 - c_r$, d_s by $1 - d_s$ and ω by $\omega\,\delta^{s-r+t-u-1}$, we obtain

$$(2.5)\qquad \omega^{\mu+y\delta} = \frac{(1 - c_r + y\delta)_{-\mu-y\delta}(\beta_u + \mu + y\delta)_{-\mu-y\delta}}{(1 - d_s + y\delta)_{-\mu-y\delta}(\alpha_t + \mu + y\delta)_{-\mu-y\delta}}\,\delta^{(\mu+y\delta)(r-s+u-t+1)}$$

$$\times \sum_{N=0}^{\infty} \frac{(2N+\nu)(-\mu-y\delta)_N}{N!\,(\nu+N)_{\mu+y\delta+1}} \cdot {}_{r+u+2}F_{s+t}\left[\begin{matrix} -N, \nu+N, 1-c_r-\mu, \beta_u; \omega\,\delta^{s-r+t-u-1} \\ \alpha_t, 1-d_s-\mu \end{matrix}\right].$$

Now to show (2. 1), expressing the H-function of both the sides of (2. 1) as a Mellin--Barnes type integral (1. 1), using (2. 5) with $\delta = 1$ in the left hand side, the equality in (2. 1) is established after some simplification.

We now discuss in brief with the help of [(16)], [(22)] and [(23)] the convergence of (2. 1) under the stated conditions.

The necessary conditions to ensure the convergence and meaning of the H-functions and the hypergeometric functions are covered in (*i*) to (*iv*). The conditions (*i*) and (*ii*) also insure that the gamma-functions in (2. 1) which appear outside are finite. The sufficient conditions to ensure the convergence of the expansion are covered by (*v*) and (*vi*). These conditions arise from the convergence of the infinite series and are based on the asymptotic behaviour for large N of the H function and the hypergeometric function on the right hand side of (2. 1). If the condition (*iv*) is not satisfied the expansion diverge on account of the analysis given in [(23)]. Even when the expansion diverges a meaning in asymptotic sense can be assigned to the series.

3. Theorem II

(i) *Let none of the following quantities be negative integers:*

$$b_m/f_m + \frac{\mu - h + 1}{h};\ b_m/f_m + \frac{\mu + \alpha_t - h}{h};\ \beta_u - 1;\ b_m/f_m - a_n/e_n,$$

where h *is a positive number.*

(ii) *Let*

$A + h(r - s) \leq 0$, $B + h(r - s) > 0$, $|\arg z| < \frac{\pi}{2}\{B + h(r - s)\}$, $Re(\mu + \alpha_t + h\,b_m/f_m) > 0$,

$Re\,\alpha_t > 0$, $Re(h\,b_m/f_m - c_r) > -1$, $Re(h\,b_m/f_m - d_s) > -1$,

$Re(1 - c_r - \mu) > 0$, $Re(1 - d_s - \mu) > 0$.

(iii) *Let* p, q, r, s, t, u, m *and* n *be positive integers or zero*;

$p+r \leqq q+s-1$ or $p+r=q+s$ and $|z\omega^h|<1$,

$p+t \leqq q+u-1$,

$0 \leqq m \leqq q$; $0 \leqq n \leqq p$; $q+s \geqq 1$.

(iv) *Let* $r+u+1=s+t$.

(v) *Let* $0<\omega<\infty, z \neq 0$.

(vi) *Let* $\Sigma d_s - \Sigma c_r + \Sigma \beta_u - \Sigma \alpha_t - 2h\, b_m/f_m < (s-r)(1-\mu)+2\mu-\dfrac{1}{2}$,

$$1+b_m/f_m-\frac{c_r+h+1}{h}>0,\ 1+b_m/f_m-\frac{h-\beta_u-\mu}{h}>0.$$

Then

$$\omega^\mu H^{m,n+r}_{p+r,q+s}\left[z\omega^h \,\middle|\, \begin{matrix}(c_r,h),(a_p,e_p)\\ b_q,f_q),(d_s,h)\end{matrix}\right]=\frac{\Gamma(1-c_r-\mu)\,\Gamma(\beta_u)}{\Gamma(1-d_s-\mu)\,\Gamma(\alpha_t)}\sum_{N=0}^{\infty}\frac{(-1)^N}{N!} \tag{3.1}$$

$$\cdot H^{m,n+t+1}_{p+t+1,q+u+1}\left[z \,\middle|\, \begin{matrix}(-\mu,h),(1-\alpha_t-\mu,h),(\alpha_p,e_p)\\ (b_q,f_q),(1-\beta_u-\mu,h),(N-\mu,h)\end{matrix}\right]$$

$$\times {}_{r+u+1}F_{s+t}\left[\begin{matrix}-N,1-c_r-\mu,\beta_u\\ \alpha_t,1-d_s-\mu\end{matrix};\omega\right].$$

PROOF: The proof of this theorem is similar to that of theorem I and is based on (1. 3) instead of (1. 2).

A formal proof follows by using the confluence principle in theorem I. That is, replacing z by λz, ω by ω/λ and making $\lambda \to \infty$.

4. Expansions for the G-function

THEOREM III

(i) *Let none of the following quantities be negative integers:*

$$b_m+\frac{\mu-i}{h};\ b_m+\frac{\mu+\alpha_t-1-i}{h};\ \nu;\ \beta_u-1;\ b_m-a_n,$$

where $i=0,1,2,\cdots,h-1$.

(ii) *Let* $(s-r)h+p+q<2(m+n), |\arg z|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q+\frac{1}{2}h(r-s)\right)\pi,$

$$Re(\mu+\alpha_t+h\,b_m)>0,\ Re(\nu-\alpha_t)>0,\ Re\,z_t>0,\ Re(h\,b_m-c_r)>-1,\ Re(h\,b_m-d_s)>-1,$$

$$Re(1-c_r-\mu)>0,\ Re(1-d_s-\mu)>0.$$

(iii) *Let* h *be a positive integer and* p, q, r, s, t, u, m *and* n *be positive integers or zero;*

$p+hr\leqq q+hs-1$ *or* $p+rh=q+sh$ *and* $|z\,\omega^h|<1,$

$p+th\leqq q+(u+1)h-1$ *or* $p+th=q+(u+1)h$ *and* $|z|>1,$

$0\leqq m\leqq q;0\leqq n\leqq p;q+s\geqq 1.$

(iv) *Let* $r+u+1=s+t.$

(v) *Let* $0<\omega<1, z\neq 0.$

(vi) *Let*

$$\Sigma\,d_s-\Sigma\,c_r+\Sigma\,\beta_u-\Sigma\,\alpha_t-2\,h\,b_m<(s-r)(1-\mu)+2\,\mu+\frac{1}{2},$$

$$1+b_m-\frac{c_r+i}{h}>0,1+b_m-\frac{1-\beta_u-\mu+i}{h}>0, i=0,1,2,\cdots,h-1.$$

Then

$$(4.1)\quad \omega^{\mu}G^{m,n+rh}_{p+rh,q+sh}\left[z\,\omega^h\left|\begin{matrix}\Delta(h,c_r),a_p\\ b_q,\Delta(h,d_s)\end{matrix}\right.\right]=\frac{\Gamma(1-c_r-\mu)\Gamma(\beta_u)}{\Gamma(1-d_s-\mu)\Gamma(\alpha_t)}(2\pi)^{\frac{1}{2}(h-1)(r+u-s-t+1)}$$

$$\cdot\, h^{\Sigma c_r-\Sigma d_s+\Sigma\alpha_t-\Sigma\beta_u+\left(\mu-\frac{1}{2}\right)(r-s)+\frac{1}{2}(u-t-1)-\nu}\times\sum_{N=0}^{\infty}\frac{(-1)^N(\nu+2N)\Gamma(\nu+N)}{N!}$$

$$\cdot\, G^{m,n+(t+1)h}_{p+(t+1)h,q+(u+2)h}\left[z\left|\begin{matrix}\Delta(h,-\mu),\Delta(h,1-\alpha_t-\mu),a_p\\ b_q,\Delta(h,N-\mu),\Delta(h,-\mu-\nu-N),\Delta(h,1-\beta_u-\mu)\end{matrix}\right.\right]$$

$$\times\,{}_{r+u+2}F_{s+t}\left[\begin{matrix}-N,\nu+N,1-c_r-\mu,\beta_u;\omega\,h^{s-r+t-u-1}\\ \alpha_t,1-d_s-\mu\end{matrix}\right].$$

PROOF: In (2.1), assuming h as a positive integer, putting $e_j=f_i=1\,(j=1,2,\cdots,p;i=1,2,\cdots,q)$, using the formula

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_p,1)\\(b_q,1)\end{matrix}\right.\right]=G^{m,n}_{p,q}\left[z\left|\begin{matrix}a_p\\b_q\end{matrix}\right.\right],$$

and simplifying with the help of (1. 1), [(17), p. 4, (11)] and [(17), p. 207, (1)], the result (4. 1) is obtained.

THEOREM IV.

(i) *Let none of the following quantities be negative integers:*

$$b_m + \frac{\mu - i}{h}\,;\, b_m + \frac{\mu + \alpha_t - 1 + i}{h}\,;\, \beta_u - 1\,;\, b_m - a_n\,,$$

where $i = 0, 1, 2, \cdots, h-1$.

(ii) *Let*

$$(s\,r) - h + p + q < 2(m+n),\ |\arg z| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q + \frac{1}{2}h(r-s)\right)\pi\,,$$

$$Re(\mu + \alpha_t + h\,b_m) > 0\,,\ Re\,\alpha_t > 0\,,\ Re(h\,b_m - c_r) > -1\,,\ Re(h\,b_m - d_s) > -1\,,$$
$$Re(1 - c_r - \mu) > 0,\ Re(1 - d_s - \mu) > 0\,.$$

(iii) *Let* h *be positive integer and* p, q, r, s, t, u, m *and* n *be positive integers or zero*;

$p + r\,h \leqq q + s\,h - 1$ *or* $p + r\,h = q + s\,h$ *and* $|z\,\omega^h| < 1$,

$p + t\,h \leqq q + u\,h - 1$,

$0 \leqq m \leqq q\,; 0 \leqq n \leqq p\,; q + s \geqq 1$.

(iv) *Let* $r + u + 1 = s + t$.

(v) *Let* $0 < \omega < \infty,\ z \neq 0$.

(vi) *Let*

$$\Sigma\,d_s - \Sigma\,c_r + \Sigma\,\beta_u - \Sigma\,\alpha_t - 2\,h\,b_m < (s-r)(1-\mu) + 2\mu - \frac{1}{2},\ 1 + b_m - \frac{c_r + i}{h} > 0\,,$$

$$1 + b_m - \frac{1 - \beta_u - \mu + i}{h} > 0,\ i = 0, 1, 2, \cdots, h-1\,.$$

Then

$$(4.\,2)\qquad \omega^{\mu}\,G^{m,\,n+rh}_{p+rh,\,q+sh}\left[z\,\omega^h \,\middle|\, \begin{matrix}\Delta(h\,,\,c_r)\,,\,a_p\\ b_q\,,\,\Delta(h\,,\,d_s)\end{matrix}\right] = \frac{\Gamma(1-c_r-\mu)\,\Gamma(\beta_u)}{\Gamma(1-d_s-\mu)\,\Gamma(\alpha_t)}\,(2\pi)^{\frac{1}{2}(h-1)(r+u-s-t)}$$

$$\cdot\, h^{\Sigma c_r - \Sigma d_s + \Sigma \alpha_t - \Sigma \beta_u + \left(\mu - \frac{1}{2}\right)(r-s+1) + \frac{1}{2}(s-t+1)} \times \sum_{N=0}^{\infty} \frac{(-1)^N\,h^N}{N!}$$

$$\cdot\, G^{m', n+(t+1)h}_{p+(t+1)h,\, q+(u+1)h}\left[z h^h \,\middle|\, \begin{matrix} \Delta(h, -\mu), \Delta(h, 1-\alpha_t-\mu), a_p \\ b_q, \Delta(h, 1-\beta_u-\mu), \Delta(h, N-\mu) \end{matrix} \right]$$

$$\times\, {}_{r+u+1}F_{s+t}\left[\begin{matrix} -N, 1-c_r-\mu, \beta_u;\ \omega h^{s-r+t-u-1} \\ \alpha_t, 1-d_s-\mu \end{matrix} \right].$$

Proof: The proof is very similar to that of theorem III and follows from theorem II by reducing the H-functions to the G-functions. A formal proof follows by applying the confluence principle in theorem III. That is, replacing z by λz, ω by ω/λ and $\lambda \to \infty$.

In view of an identity, which is apparent from the definition of the G-function, viz.

$$G^{m, n+(t+1)h}_{p+(t+1)h,\, q+(u+1)h}\left[z \,\middle|\, \begin{matrix} \Delta(h, -\mu), \Delta(h, 1-\alpha_t-\mu), a_p \\ b_q, \Delta(h, 1-\beta_u-\mu), \Delta(h, N-\mu) \end{matrix} \right]$$

$$= (-1)^N G^{m+h, n+th}_{p+(t+1)h,\, q+(u+1)h}\left[z \,\middle|\, \begin{matrix} \Delta(h, 1-\alpha_t-\mu), a_p, \Delta(h, -\mu) \\ \Delta(h, N-\mu), \Delta(h, 1-\beta_u-\mu) \end{matrix} \right],$$

the theorem IV reduces to a result recently given by the author [(13), (3. 7)].

5. Expansions for the *E*-function

Theorem V

(i) *Let none of the following quantities be negative integers:*

$$\frac{\mu-i}{h};\ \frac{\mu+\alpha_t-1-i}{h};\ \nu;\ \beta_u-1;\ a_n-1,$$

where $i = 0, 1, 2, \cdots, h-1$.

(ii) *Let*

$$q+hs < p+hr+1,\ |\arg z| < (p-q+hr-hs+1)\frac{\pi}{2},\ Re(\mu+\alpha_t) > 0,$$

$$Re(\nu-\alpha_t) > 0,\ Re\,\alpha_t > 0,\ Re\,c_r > 0,\ Re\,d_s > 0,\ Re(c_r-\mu) > 0,\ Re(d_s-\mu) > 0.$$

(iii) *Let* h *be a positive integer and* p, q, r, s, t *and* u *be positive integers or zero;*

$p+rh < q+sh$ *or* $p+rh = q+sh+1$ *and* $|z\omega^h| > 1$,

$p+th < q+(u+1)h$ *or* $p+th = q+(u+1)h+1$ *and* $|z| > 1$,

$0 \leqq q;\ 0 \leqq p;\ q+s \geqq 0.$

(iv) *Let* $r + u + 1 = s + t$.

(v) *Let* $1 < \omega < \infty$.

(vi) *Let*

$$\Sigma c_r - \Sigma d_s + \Sigma \beta_u - \Sigma \alpha_t < (r - s)\mu + 2\mu + \frac{1}{2}, \quad 1 - \frac{1 - c_r + i}{h} > 0,$$

$$1 - \frac{1 - \beta_u - \mu + i}{h} > 0, \quad i = 0, 1, 2, \cdots, h - 1.$$

Then

$$(5.1) \qquad \omega^{-\mu} E\left[\begin{matrix} a_p, \Delta(h, c_r) : z\,\omega^h \\ b_q, \Delta(h, d_s) \end{matrix}\right] = \frac{\Gamma(c_r - \mu)\Gamma(\beta_u)}{\Gamma(d_s - \mu)\Gamma(\alpha_t)} \times (2\pi)^{\frac{1}{2}(h-1)(r+u-s-t+1)}$$

$$\cdot\, h^{\Sigma d_s - \Sigma c_r + \Sigma \alpha_t - \Sigma \beta_u + \left(\mu + \frac{1}{2}\right)(r - s) + \frac{1}{2}(u - t - 1) - \nu} \times \sum_{N=0}^{\infty} \frac{(-1)^N(\nu + 2N)\Gamma(\nu + N)}{N!}$$

$$\cdot\, E\left[\begin{matrix} a_p, \Delta(h, \mu + 1), \Delta(h, \alpha_t + \mu) : z \\ b_q, \Delta(h, 1 + \mu - N), \Delta(h, 1 + \mu + \nu + N), \Delta(h, \beta_u + \mu) \end{matrix}\right]$$

$$\times\, {}_{r+u+2}F_{s+t}\left[\begin{matrix} -N, \nu + N, c_r - \mu, \beta_u;\ \omega^{-1} h^{s-r+t-u-1} \\ \alpha_t, d_s - \mu \end{matrix}\right].$$

PROOF: In (4.1), putting $m = 1$, $n = p$, $b_1 = 0$, replacing q by $q + 1$ and b_{j+1} by $b_j (j = 1, 2, \cdots, q)$, using [(17), p. 209, (9)] and [(19), p. 444, (2)] replacing z by z^{-1}, ω by ω^{-1}, $1 - a_p$ by a_p, $1 - b_q$ by b_q, $1 - c_r$ by c_r and $1 - d_s$ by d_s the result (5.1) is obtained.

THEOREM VI

(i) *Let none of the following quantities be negative integers:*

$$\frac{\mu - i}{h}; \quad \frac{\mu + \alpha_t - 1 + i}{h}; \quad \beta_u - 1; \quad a_n - 1,$$

where $i = 0, 1, 2, \cdots, h - 1$.

(ii) *Let*

$$q + hs < p + hr + 1, \quad |\arg z| < (p - q + hr - hs + 1)\pi/2, \quad Re(\mu + \alpha_t) > 0,$$

$$Re\,\alpha_t > 0, \quad Re\,c_r > 0, \quad Re\,d^s > 0, \quad Re(c_r - \mu) > 0, \quad Re(d_s - \mu) > 0.$$

(iii) *Let* h *be a positive integer and* p, q, r, s, t *and* u *be positive integers or zero.*

$p + rh \leqq q + sh$ *or* $p + rh = q + sh + 1$,

$p + th \leqq q + uh$,

$0 \leqq q$; $0 \leqq p$; $q + s \geqq 0$.

(iv) *Let* $r + u + 1 = s + t$.

(v) *Let* $0 < \omega < \infty$.

(vi) *Let* $\Sigma c_r - \Sigma d_s + \Sigma \beta_u - \Sigma \alpha_t < (r - s)\mu + 2\mu - \frac{1}{2}, 1 - \frac{1 - c_r + i}{h} > 0$,

$$1 - \frac{1 - \beta_u - \mu + i}{h} > 0, i = 0, 1, 2, \cdots, h - 1.$$

$$(5.\,2) \qquad \omega^{-\mu} E\begin{bmatrix} a_p, \Delta(h, c_r) & : z\omega^h \\ b_q, \Delta(h, d_s) & \end{bmatrix} = \frac{\Gamma(c_r - \mu)\Gamma(\beta_u)}{\Gamma(d_s - \mu)\Gamma(\alpha_t)} \times (2\pi)^{\frac{1}{2}(h-1)(r+u-s-t)}$$

$$\cdot h^{\Sigma d_s - \Sigma c_r + \Sigma \alpha_t - \Sigma \beta_u + \left(\mu + \frac{1}{2}\right)(r-s) + \mu + \frac{1}{2}(s-t)} \times \sum_{N=0}^{\infty} \frac{(-1)^N h^N}{N!}$$

$$\cdot E\begin{bmatrix} a_p, \Delta(h, 1+\mu), \Delta(h, \alpha_t + \mu) & : z h^h \\ b_q, \Delta(h, 1 + \mu - N), \Delta(h, \beta_u + \mu) & \end{bmatrix}$$

$$\times {}_{r+u+1}F_{s+t}\begin{bmatrix} -N, c_r - \mu, \beta_u ; \omega^{-1} h^{s-r+t-u+1} \\ \alpha_t, d_s - \mu \end{bmatrix}.$$

Proof: In (4. 2), reducing the G-function to the E-function as in theorem V, we get the formula (5. 2).

6. Particular cases:

Since the H-function can be reduced to the G-function, which itself is a generalization of many higher transcendental functions [(17), pp. 215-222], therefore the theorems given here are of general character and hence may encompass several cases of interest. However, a few well known particular cases are mentioned below:

(*i*) In (2. 1), putting $e_j = f_j = 1\,(j = 1, 2, \cdots, p; i = 1, 2, \cdots, q)\,h = 1, b_1 = 0$, $m = 1, n = p$, replacing a_p by $1 - a_p, q$ by $q + 1$ and b_{j+1} by $1 - b_j (j = 1, 2, \cdots, q)$ c_r by $1 - c_r, d_s$ by $1 - d_s$, using the formula (2. 3) and simplifying we get a known result given by Wimp and Luke [(42), p. 353, (1. 6)].

(*ii*) In (3. 1), reducing the H-function to the hypergeometric function as above, we obtain another result given by WIMP and LUKE [(42), p. 358, (1. 18)].

(*iii*) In (4. 1), taking $h = 1$, $\mu = 0$, it reduces to a known result [(42), p. 359, (2. 2)].

(*iv*) In (4. 2), setting $h = 1$, $\mu = 0$, we get a known result [(42), p. 360, (2. 3)].

(*v*) In (4. 2), substituting $h = 1$, $\mu = 0$, $t = u = 0$, $s = r$, we obtain a known result [(32), p. 43, (51)], which is one of the most general expansions given by MEIJER.

I am thankful to Principal Dr. S. M. DAS GUPTA for the facilities he provided to me.

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