A theorem on abelian quotient groups of a group

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1. In a previous paper [1], we have shown that, if α is a normal endomorphism of the group G, then the operator $\alpha - \alpha^2$ is also a normal endomorphism of G and the quotient group $G/\mathrm{Ker}(\alpha - \alpha^2)$ is an abelian group.

In this note, we extend this result; we state a necessary and sufficient condition in order that the quotient group $G/\mathrm{Ker}(\beta - \beta\alpha)$ be an abelian group, α and β being endomorphisms of G and $\beta\alpha$ being the composite of β and α.

2. It is well known that, if α and β are endomorphisms of the group G, then the operator $\beta - \alpha$ of G, defined by

$$(\beta - \alpha)(x) = \beta(x)\,\alpha(x^{-1}) \text{ for every } x \in G$$

need not be an endomorphism of G.

In fact, since

$$(\beta - \alpha)(xy) = \beta(xy)\,\alpha(y^{-1}x^{-1}) = \\ = \beta(x)\,\beta(y)\,\alpha(y^{-1})\,\alpha(x^{-1})$$

and, on the other hand,

$$(\beta - \alpha)(x)\,(\beta - \alpha)(y) = \beta(x)\,\alpha(x^{-1})\,\beta(y)\,\alpha(y^{-1}),$$

one concludes that $\beta - \alpha$ is an endomorphism, if and only if one has

$$\beta(y)\,\alpha(y^{-1})\,\alpha(x^{-1}) = \alpha(x^{-1})\,\beta(y)\,\alpha(y^{-1})$$

for all x, y in G.

This means that the following holds:

Lemma. *If α and β are endomorphisms of the group G, then the operator $\beta - \alpha$ is an endomorphism of G, if and only if the image of $\beta - \alpha$ is in the centralizer of the image of α in G.*

Theorem. *Let α and β be endomorphisms of the group G. Then the operator $\beta - \beta\alpha$ is an endomorphism of G and the quotient group $G/Ker(\beta - \beta\alpha)$ is abelian, if and only if the image of $\beta - \beta\alpha$ is contained in the center of the image of β.*

Proof. Let $\beta - \beta\alpha$ be an endomorphism. Then, as it is well known, the quotient group $G/\mathrm{Ker}(\beta - \beta\alpha)$ is abelian, if and only if the kernel of the endomorphism $\beta - \beta\alpha$ contains the commutator subgroup of G, that is to say,

$$(1) \qquad (\beta - \beta\alpha)(x y x^{-1} y^{-1}) = e$$

for all x, y in G, e being the neutral element of G.

First, let us suppose that one has

$$(2) \qquad Im(\beta - \beta\alpha) \subseteq Center\ of\ Im(\beta).$$

Since

$$Center\ of\ Im(\beta) \subseteq Centralizer\ of\ Im(\beta\alpha)\ in\ G,$$

one concludes by Lemma above that the operator $\beta - \beta\alpha$ is an endomorphism of G.

Furthermore, one has

$$\begin{aligned}
&(\beta - \beta\alpha)(x y x^{-1} y^{-1}) = \\
&= \beta(x y x^{-1} y^{-1})\,\beta\alpha(x y x^{-1} y^{-1})^{-1} = \\
&= \beta(x)\,\beta(y)\,\beta(x^{-1})\,\beta(y^{-1}) \cdot \\
&\cdot \beta\alpha(y)\,\beta\alpha(x)\,\beta\alpha(y^{-1})\,\beta\alpha(x^{-1}) = \\
&= \beta(x)\,\beta(y)\,\beta(y^{-1})\,\beta\alpha(y)\,\beta(x^{-1}) \cdot \\
&\cdot \beta\alpha(x)\,\beta\alpha(y^{-1})\,\beta\alpha(x^{-1}) = \\
&= \beta(x)\,\beta(x^{-1})\,\beta\alpha(x)\,\beta\alpha(y) \cdot \\
&\cdot \beta\alpha(y^{-1})\,\beta\alpha(x^{-1}) = e.
\end{aligned}$$

proving (1).

Now, let us suppose that the operator $\beta - \beta\alpha$ is an endomorphism of the group G such that the quotient group $G/\mathrm{Ker}(\beta - \beta\alpha)$ is abelian.

From (1) it follows

$$(\beta - \beta\alpha)(xy)(\beta - \beta\alpha)(x^{-1}y^{-1}) = e,$$

hence

$$(\beta - \beta\alpha)(x)(\beta - \beta\alpha)(y) = (\beta - \beta\alpha)(yx),$$

that is to say,

$$\beta(x)\beta\alpha(x^{-1})\beta(y)\beta\alpha(y^{-1}) = \beta(yx)\beta\alpha((yx)^{-1}).$$

Consequently,

$$\beta(x)\beta\alpha(x^{-1})\beta(y)\beta\alpha(y^{-1}) = \\ = \beta(y)\beta(x)\beta\alpha(x^{-1})\beta\alpha(y^{-1})$$

and so

$$\beta(x)\beta\alpha(x^{-1})\beta(y) = \beta(y)\beta(x)\beta\alpha(x^{-1})$$

for all x, y in G.

Thus, for every $x \in G$, the element $\beta(x)\beta\alpha(x^{-1})$ commutes with every element of $\mathrm{Im}(\beta)$, that is to say,

$$\mathrm{Im}(\beta - \beta\alpha) \subseteq \textit{Center of } \mathrm{Im}(\beta),$$

as wanted.

3. In particular, let us set $\beta = \varepsilon$ (identity operator).

One has clearly $\mathrm{Im}(\varepsilon) = G$ and, since the condition

$$\mathrm{Im}(\varepsilon - \varepsilon\alpha) \subseteq \textit{Center of } G$$

means that

$$x\alpha(x^{-1}) \in \textit{Center of } G$$

for every $x \in G$, one obtains

COROLLARY 1. *If α is an endomorphism of G, then the quotient group $G/Ker(\varepsilon - \alpha)$ is abelian, if and only if, for every $x \in G$, $x\alpha(x^{-1})$ is in the center of G.*

This Corollary is the Theorem 3 in [1].

Now, let us set $\beta = \alpha$.

Then, if the endomorphism α is normal, i. e., if

$$(3) \quad \alpha(uvu^{-1}) = u\alpha(v)u^{-1} \text{ for all } u, v \text{ in } G,$$

one sees that the condition (2) holds.

Indeed, from (3) it follows, by setting $u = \alpha(x^{-1})$ and $v = y$,

$$\alpha^2(x^{-1})\alpha(y)\alpha^2(x) = \alpha(x^{-1})\alpha(y)\alpha(x)$$

that is to say,

$$\alpha(x)\alpha^2(x^{-1})\alpha(y) = \alpha(y)\alpha(x)\alpha^2(x^{-1})$$

for all x, y in G.

This means that

$$(\alpha - \alpha^2)(x)\alpha(y) = \alpha(y)(\alpha - \alpha^2)(x) \\ \text{for all } x, y \text{ in } G$$

and so the condition (2) holds.

By Theorem above, the group $G/\mathrm{Ker}(\alpha - \alpha^2)$ is abelian and one obtains the following result, stated in [1] as Theorem 2:

COROLLARY 2. *If α is a normal endomorphism of G, then $\alpha - \alpha^2$ is also an endomorphism and $G/Ker(\alpha - \alpha^2)$ is abelian.*

BIBLIOGRAPHY

[1] JOSÉ MORGADO, *A note on the normal endomorphisms of a group,* «Gazeta de Matemática», n.º 109-112, 1968, pag. 6-8.

[2] H. J. ZASSENHAUS, *The Theory of Groups,* second edition, Chelsea Publishing Company, New York, 1958.