Cardinality of a set of topologies

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Let E be an infinite set and K be the set of all topologies τ on E, such that (E,τ) is a Hausdorff compact connected and locally connected space. Question: Which is the cardinal number of the set K? The purpose of this note is to answer this question.

1. For any set Z, $|Z|$ denotes the cardinal number of Z.

Suppose that K is a nonempty set and let τ be a fixed element of K. Since (E,τ) is a normal connected space, the cardinality of the set E is greater than or equal to $2^{\aleph_0}$.

Now, we fix an element $b \in E$. Let us denote by τ' the product topology on $[0,1]\times E$, where E is topologized with τ and $[0,1]$ with the usual topology. Put $G=[0,1]\times E - \{(0,b)\}$. G as a subspace of $[0,1]\times E$ is a Hausdorff locally compact connected and locally connected space.

For each partition $\{X_1, X_2\}$ of $E-\{b\}$ (thus $E-\{b\}=X_1\cup X_2$ and $X_1\cap X_2=\emptyset$), with $|X_1|=|X_2|=|E|$, we fix two bijective functions $f_1: G\to X_1$ and $f_2: G \to X_2$. (The set of all partitions of $E-\{b\}$ under the above conditions has cardinality equal to $2^{|E|}$ by virtue of [1] and because $|E\times E|=|E|$). Putting $P=\{X_1, X_2\}$, let us denote by τ_P te topology on E such that the set

$$\{f_1(Z\cap G)\cup f_2(Z\cap G)\cup\{b\} \mid (0,b)\in Z\in\tau'\}\cup$$
$$\cup\{f_1(Z\cap G)\mid Z\in\tau'\}\cup\{f_2(Z\cap G)\mid Z\in\tau'\}$$

is an open basis of the topology τ_P.

Proposition 1. (E,τ_P) *is a* Hausdorff *compact connected and locally connected space.*

Proof. It is obvious that (E,τ_P) is a Hausdorff space. On the other hand, X_1 and X_2 are open-connected subsets and it follows immediately that (E,τ_P) is connected. The subspaces $X_1\cup\{b\}$ and $X_2\cup\{b\}$ are compact; thus (E,τ_P) is a compact space.

Finally, it is sufficient to prove that b has a fundamental system of connected neighborhoods. Let W be a connected neighborhood of b in the topological space (E,τ) and put $Z=[0,1/n[\times W$, where $n \geq 1$ is a natural number. The set Z is a connected neighborhood of $(0,b)$ in the product topological space $([0,1]\times E,\tau')$. Therefore $f_1(Z\cap G)\cup \cup f_2(Z\cap G)\cup\{b\}$ is a connected neighborhood of b in the topological space (E,τ_P). So it follows easily that (E,τ_P) is locally connected.

2. Let $Q=\{Y_1, Y_2\}$ be another partition of $E-\{b\}$, with $|Y_1|=|Y_2|=|E|$, and let τ_Q be the correspondent topology on E. We shall prove that if $P\neq Q$, then $\tau_P\neq\tau_Q$.

On the contrary, suppose that $P\neq Q$ and $\tau_P=\tau_Q$. The sets X_1, X_2, Y_1 and Y_2 are open-connected in (E,τ_P) and $X_1\cup X_2 = Y_1\cup Y_2$. Thus $X_1=Y_1$ and $X_2=Y_2$ or $X_1=Y_2$ and $Y_1=X_2$. It follows that $P=Q$, which is impossible.

Proposition 2. *The cardinality of the set* K *is equal to* 0 *or to* $2^{|E|}$.

Proof. Since for any $\sigma\in K, (E,\sigma)$ is Hausdorff compact, then $|K|$ is less than or equal to $2^{|E|}$. So the proof is completed.

REFERENCES

[1] E. Farah, *Number of equivalence relations on a set.* Ciência e Cultura **18** (1966), n.º 4, p. 437.