Fourier series for G-function of two variables

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1. Introduction. The object of this paper is to establish four integrals involving G-function of two variables and use them to evaluate four FOURIER series for G-function of two variables. Some results given by KESARWANI [5] and BAJPAI [2] are shown as particular cases.

The symbol $\Delta(\delta, \alpha)$ represents the set of parameters $\alpha/\delta, (\alpha+1)/\delta, \cdots, (\alpha+\delta-1)/\delta$ where δ is a positive integer and (a_p) stands for $a_1, a_2, a_3, \cdots, a_p$ throughout this paper.

AGARWAL [1] and SHARMA [8] defined the G-function of two variables in the form of MELLIN-BARNES type integral which has been symbolically denoted by BAJPAI [3, (1. 1)] as

(1. 1)
$$G^{(m_1, m_2); (n_1, n_2), n_3}_{(p_1, p_2), p_3; (q_1, q_2), q_3}\left[\begin{array}{c|l} x & (a_{p_1}); (c_{p_2}) \\ & (e_{p_3}) \\ y & (b_{q_1}); (d_{q_2}) \\ & (f_{q_3}) \end{array}\right]$$

$$= \frac{1}{(2\pi i)^2} \int_{L_1}\int_{L_2} \frac{\prod_{j=1}^{m_1}\Gamma(b_j - s)\prod_{j=1}^{n_1}\Gamma(1-a_j+s)\prod_{j=1}^{m_2}\Gamma(d_j-t)\prod_{j=1}^{n_2}\Gamma(1-c_j+t)\prod_{j=1}^{n_3}\Gamma(1-e_j+s+t)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j+s)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-s)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+t)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-t)} \times$$

$$\times \frac{x^s y^t}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j - s - t)\prod_{j=1}^{q_3}\Gamma(1-f_j+s+t)}\, ds\, dt.$$

The contour L_1 is in the s-plane and runs from $-i\infty$ to $+i\infty$ with loops if necessary, to ensure that the poles of $\Gamma(b_j - s), j = 1, 2, \cdots, m_1$ lie on the right and the poles of $\Gamma(1-a_j+s), j = 1, 2, \cdots, n_1$ and $\Gamma(1-e_j+s+t), j = 1, \cdots, n_3$ to the left of the contour. Similarly the contour L_2 is in the t-plane and runs from $-i\infty$ to $+i\infty$ with loops if necessary, to ensure that the poles of $\Gamma(d_j - t), j = 1, 2, \cdots, m_2$ lie on the right and the poles of $\Gamma(1-c_j+t), j = 1, 2, \cdots, n_2$ and $\Gamma(1-e_j+s+t), j = 1, 2, \cdots, n_3$ lie on the left of the contour.

Provided that $0 \leq m_1 \leq q_1$, $0 \leq m_2 \leq q_2$, $0 \leq n_1 \leq p_1$, $0 \leq n_2 \leq p_2$, $0 \leq n_3 \leq p_3$; the integral converges if

$$(1.\,2)\qquad \begin{cases} (p_3+q_3+p_1+q_1) < 2\,(m_1+n_1+n_3)\,;\,(p_3+q_3+p_2+q_2) < 2\,(m_2+n_2+n_3)\,; \\ |\arg x| < \left[m_1+n_1+n_3-\dfrac{1}{2}(p_1+q_1+p_3+q_3)\right]\pi\,, \\ |\arg y| < \left[m_2+n_2+n_3-\dfrac{1}{2}(p_2+q_2+p_3+q_3)\right]\pi\,. \end{cases}$$

The following formulae are required in the proofs.

$$(1.\,3)\qquad \int_0^\pi \sin(2\,n+1)\,\theta\,(\sin\theta)^{1-2\zeta}\,d\,\theta = \frac{\sqrt{\pi}\,\Gamma(3/2-\zeta)\,\Gamma(n+\zeta)}{\Gamma\zeta\,\Gamma(2-\zeta+n)}\ \mathrm{Re}\,(3-\zeta)>0\,,\,n=0\,,1\,,\cdots$$

which follows from [(7), p. 80]

$$(1.\,4)\qquad \int_0^\pi \cos n\,\theta\,(\sin\theta/2)^{-2\zeta}\,d\,\theta = \frac{\sqrt{\pi}\,\Gamma(\zeta+n)\,\Gamma(1/2-\zeta)}{\Gamma\zeta\,\Gamma(1-\zeta+n)}\ \mathrm{Re}(1-2\zeta)>0,\,n=0,1,2,\cdots$$

which follows from [6, p. 143]

2. The integrals to be evaluated are

$$(2.\,1)\qquad \int_0^\pi \sin(2\,u+1)\,\theta\,(\sin\theta)^{1-2\zeta}\,G^{(m_1,m_2);(n_1,n_2),n_3}_{(p_1,p_2),p_3;(q_1,q_2),q_3}\left[\begin{matrix} x\,(\sin\theta)^{-2\delta} \\ \\ y \\ \\ \end{matrix}\left|\begin{matrix}(a_{p_1});(c_{p_2}) \\ (e_{p_3}) \\ (b_{q_1});(d_{q_2}) \\ (f_{q_3})\end{matrix}\right.\right]d\,\theta$$

$$= \sqrt{\frac{\pi}{\delta}}\,G^{(m_1+\delta,m_2);(n_1+\delta,n_2),n_3}_{(p_1+2\delta,p_2),p_3;(q_1+2\delta,q_2),q_3}\left[\begin{matrix} x \\ \\ y \\ \\ \end{matrix}\left|\begin{matrix}\Delta(\delta\,,1-\zeta-u)\,,(a_{p_1})\,,\Delta(\delta\,,2-\zeta+u)\,;(c_{p_2}) \\ (e_{p_3}) \\ \Delta\left(\delta\,,\dfrac{3}{2}-\zeta\right),(b_{q_1})\,,\Delta(\delta\,,1-\zeta)\,;(d_{q_2}) \\ (f_{q_3})\end{matrix}\right.\right]$$

where

$$\mathrm{Re}\,[3-2\,\zeta+2\,\delta\,(1-a_j)]>0\,,\,j=1\,,2\,,\cdots n_1\,;\,u=0\,,1\,,2\,,\cdots.$$

Other conditions of validity being the same as (1. 2)

$$(2.\,2)\qquad \int_0^\pi \sin(2\,u+1)\,\theta\,(\sin\theta)^{1-2\zeta}\,G^{(m_1,m_2);(n_1,n_2),n_3}_{(p_1,p_2),p_3;(q_1,q_2),q_3}\left[\begin{matrix} x \\ \\ y\,(\sin\theta)^{-2\delta} \\ \\ \end{matrix}\left|\begin{matrix}(a_{p_1});(c_{p_2}) \\ (e_{p_3}) \\ (b_{q_1});(d_{q_2}) \\ (f_{q_3})\end{matrix}\right.\right]d\theta$$

$$= \sqrt{\frac{\pi}{\partial}}\, G^{(m_1, m_2+\delta);(n_1, n_2+\delta), n_3}_{(p_1, p_2+2\delta), p_3;(q_1, q_2+2\delta), q_3} \left[\begin{array}{c|l} x & (a_{p_1}); \Delta(\partial, 1-\zeta-u), (c_{p_2}), \Delta(\partial, 2-\zeta+u) \\ & (e_{p_3}) \\ y & (b_{q_1}); \Delta(\partial, 3/2-\zeta), (d_{q_2}), \Delta(\partial, 1-\zeta) \\ & (f_{q_3}) \end{array} \right]$$

where

$$\operatorname{Re}[3 - 2\zeta + 2\partial(1 - c_j)] > 0, \; j = 1, 2, \cdots, n_2; \; u = 0, 1, \cdots.$$

Other conditions of validity are same as (1. 2)

$$(2.3) \qquad \int_0^{\pi} \cos u\theta\, (\sin \theta/2)^{-2\zeta}\, G^{(m_1, m_2);(n_1, n_2), n_3}_{(p_1, p_2), p_3;(q_1, q_2), q_3} \left[\begin{array}{c|l} x(\sin\theta/2)^{-2\delta} & (a_{p_1}); (c_{p_2}) \\ & (e_{p_3}) \\ y & (b_{q_1}); (d_{q_2}) \\ & (f_{q_3}) \end{array} \right] d\theta$$

$$= \sqrt{\frac{\pi}{\partial}}\, G^{(m_1+\delta, m_2);(n_1+\delta, n_2), n_3}_{(p_1+2\delta, p_2), p_3;(q_1+2\delta, q_2), q_3} \left[\begin{array}{c|l} x & \Delta(\partial, 1-\zeta-u), (a_{p_1}), \Delta(\partial, 1-\zeta+u); (c_{p_2}) \\ & (e_{p_3}) \\ y & \Delta(\partial, 1/2-\zeta), (b_{q_1}), \Delta(\partial, 1-\zeta); (d_{q_2}) \\ & (f_{q_3}) \end{array} \right]$$

where

$$\operatorname{Re}[1 - 2\zeta + 2\partial(1 - a_j)] > 0, \; j = 1, 2, \cdots, n_1; \; u = 0, 1, 2, \cdots.$$

Other conditions of validity are same as (1. 2)

$$(2.4) \qquad \int_0^{\pi} \cos u\theta\, (\sin \theta/2)^{-2\zeta}\, G^{(m_1, m_2);(n_1, n_2), n_3}_{(p_1, p_2), p_3;(q_1, q_2), q_3} \left[\begin{array}{c|l} x & (a_{p_1}); (c_{p_2}) \\ & (e_{p_3}) \\ y(\sin\theta/2)^{-2\delta} & (b_{q_1}); (d_{q_2}) \\ & (f_{q_3}) \end{array} \right] d\theta =$$

$$= \sqrt{\frac{\pi}{\partial}}\, G^{(m_1, m_2+\delta);(n_1, n_2+\delta), n_3}_{(p_1, p_2+2\delta), p_3;(q_1, q_2+\delta), q_3} \left[\begin{array}{c|l} x & (a_{p_1}); \Delta(\partial, 1-\zeta-u), (c_{p_2}), \Delta(\partial, 1-\zeta+u) \\ & (e_{p_3}) \\ y & (b_{q_1}); \Delta(\partial, 1/2-\zeta), (d_{q_2}), \Delta(\partial, 1-\zeta) \\ & (f_{q_3}) \end{array} \right]$$

where $\operatorname{Re}[1 - 2\zeta + 2\partial(1 - c_j)] > 0, \; j = 1, 2, \cdots, n_2; \;\; u = 0, 1, 2, \cdots.$
Other conditions of validity are same as (1. 2).

PROOF. To prove (2. 1), expressing the G-function as MELLIN-BARNES type integral (1. 1), interchanging the order of integration which is justified due to the absolnte convergence of the integrals involved in the process and evaluating the inner integral with the help of (1. 3), we get

$$\frac{1}{(2\pi i)^2}\int_L\int_{L_1}\frac{\prod_{j=1}^{m_1}\Gamma(b_j-s)\prod_{j=1}^{n_1}\Gamma(1-a_j+s)\prod_{j=1}^{m_2}\Gamma(d_j+t)\prod_{j=1}^{n_2}\Gamma(1-c_j+t)\prod_{j=1}^{n_3}\Gamma(1-e_j+s+t)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j+s)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-s)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+t)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-t)}\times$$

$$\times\frac{\sqrt{\pi}\,\Gamma(3/2-\zeta-\delta s)\,\Gamma(\zeta+u+\delta s)}{\prod_{=n_3+1}^{p_3}\Gamma(e_j-s-t)\prod_{j=1}^{q_3}\Gamma(1-f_j+s+t)\,\Gamma(\zeta+\delta s)\,\Gamma(2-\zeta+u-\delta s}\,x^s\,y^t\,ds\,dt\,.$$

On applying multiplication formula for Gamma functions [4, p. 4 (11)] we get

$$\sqrt{\frac{\pi}{\delta}}\,\frac{1}{(2\pi i)^2}\,.$$

$$\int_{L_1}\int_{L_2}\frac{\prod_{j=1}^{m_1}\Gamma(b_j-s)\prod_{j=1}^{n_1}\Gamma(1-a_j+s)\prod_{j=1}^{m_1}\Gamma(d_j-t)\prod_{j=1}^{n_2}\Gamma(1-c_j+t)\prod_{j=1}^{n_3}(1-e_j+s+t)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-b_j+s)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-s)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+t)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-s-t)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-t)}\times$$

$$\times\frac{\prod_{i=0}^{\delta-1}\Gamma\left(\frac{3/2-\zeta+i}{\delta}-s\right)\prod_{i=0}^{\delta-1}\Gamma\left(\frac{\zeta+u+i}{\delta}+s\right)}{\prod_{j=1}^{q_3}\Gamma(1-f_j+s+t)\prod_{i=0}^{\delta-1}\Gamma\left(\frac{\zeta+i}{\delta}+s\right)\prod_{i=0}^{\delta-1}\Gamma\left(\frac{2-\zeta+u+i)}{\delta}-s\right)}\,x^s\,y^t\,ds\,dt\,.$$

Now on using (1. 1), the integral (2. 1) is proved.

The integral (2. 2) is proved by the same method as above. The integrals (2. 3) and (2. 4) are proved on adopting the same proceedure and using (1. 4)

3. *Particular cases:* (i) If ζ is an integer, (2. 3) and (2. 4) can be written, respectively, as

$$(3.\ 1)\qquad \int_0^{\pi}(\sin\Phi)^{-2\zeta}\cos 2u\Phi\, G^{(m_1,m_2);(n_1,n_2),n_3}_{(p_1,p_2),p_3;(q_1,q_2),q_3}\left[\begin{array}{c|l} x(\sin\Phi)^{-2\delta} & (a_{p_1});(c_{p_2}) \\ & (e_{p_3}) \\ y & (b_{q_1});(d_{q_2}) \\ & (f_{q_3})\end{array}\right]d\Phi$$

$$=\sqrt{\frac{\pi}{\delta}}\;G^{(m_1+\delta,m_2);(n_1+\delta,n_2),n_3}_{(p_1+2\delta,p_2),p_3;(q_1+2\delta,q_2),q_3}\left[\begin{array}{c|l} x & \Delta(\delta,1-\zeta-u),(a_{p_1}),\Delta(\delta,1-\zeta+u);(c_{p_2}) \\ & (e_{p_3}) \\ y & \Delta(\delta,1/2-\zeta),(b_{q_1}),\Delta(\delta,1-\zeta);(d_{q_2}) \\ & (f_{q_3})\end{array}\right]$$

the conditions of validity are same as for (2. 3)

$$(3.\,2)\qquad \int_0^{\pi} (\sin\Phi)^{-2\zeta} \cos 2u\Phi \, G^{(m_1,m_2);(n_1,n_2),n_3}_{(p_1,p_2),p_3;(q_1,q_2),q_3}\left[\begin{array}{c|l} x & (a_{p_1});(c_{p_2}) \\ & (e_{p_3}) \\ y(\sin\Phi)^{-2\delta} & (b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array}\right] d\Phi$$

$$= \sqrt{\frac{\pi}{\partial}}\; G^{(m_1,m_2+\delta);(n_1,n_2+\delta)n_3}_{(p_1,p_2+2\delta),p_3;(q_1,q_2+2\partial),q_3}\left[\begin{array}{c|l} x & (a_{p_1});\Delta(\partial,1-\zeta-u),(c_{p_2}),\Delta(\partial,1-\zeta+u) \\ & (e_{p_3}) \\ y & (b_{q_1});\Delta(\partial,1/2-\zeta),(d_{q_2}),\Delta(\partial,1-\zeta) \\ & (f_{q_3}) \end{array}\right]$$

The conditions of validity are same as for (2. 4)

(*ii*) Putting $\sin(2u+1)\theta \;\sin\theta = \frac{1}{2}[\cos 2u\theta - \cos 2(u+1)\theta]$ in (2. 1) and using (3. 1), we get

$$(3.\,3)\qquad 2\,G^{(m_1+\delta,m_2);(n_1+\delta,n_2);n_3}_{(p_1+2\delta,p_2),p_3;(q_1+2\delta,q_2),q_3}\left[\begin{array}{c|l} x & \Delta(\partial,1-\zeta-u),(a_{p_1}),\Delta(\partial,2-\zeta+u);(c_{p_2}) \\ & (e_{p_3}) \\ y & \Delta(\partial,3/2-\zeta),(b_{q_1}),\Delta(\partial,1-\zeta);(d_{q_2}) \\ & (f_{q_3}) \end{array}\right]$$

$$= G^{(m_1+\delta,m_2);(n_1+\delta,n_2),n_3}_{(p_1+2\delta,p_2),p_3;(q_1+2\delta,q_2),q_3}\left[\begin{array}{c|l} x & \Delta(\partial,1-\zeta-u),(a_{p_1}),\Delta(\partial,1-\zeta+u);(c_{p_2}) \\ & (e_{p_3}) \\ y & \Delta(\partial,1/2-\zeta),(b_{q_1}),\Delta(\partial,1-\zeta);;(d_{q_2}) \\ & (f_{q_3}) \end{array}\right] -$$

$$- G^{(m_1+\delta,m_2);(n_1+\delta,n_2),n_3}_{(p_1+2\delta,p_2),p_3;(q_1+2\delta,q_2),q_3}\left[\begin{array}{c|l} x & \Delta(\partial,-\zeta-u),(a_{p_1}),\Delta(\partial,2-\zeta+u);(c_{p_2}) \\ & (e_{p_3}) \\ y & \Delta(\partial,1/2-\zeta),(b_{q_1}),\Delta(\partial,1-\zeta);(d_{q_2}) \\ & (f_{q_3}) \end{array}\right].$$

Similarly using (2. 2) and (2. 3), we obtain,

$$(3.\,4)\qquad 2\,G^{(m_1,m_2+\delta);(n_1,n_2+\delta),n_3}_{(p_1,p_2+2\delta),p_3;(q_1,q_2+2\delta),q_3}\left[\begin{array}{c|l} x & (a_{p_1});\Delta(\partial,1-\zeta-u),(c_{p_2}),\Delta(\partial,2-\zeta+u) \\ & (e_{p_3}) \\ y & (b_{q_1});\Delta(\partial,3/2-\zeta),(d_{q_2}),\Delta(\partial,1-\zeta) \\ & (f_{q_3}) \end{array}\right]$$

$$= G^{(m_1, m_2+\delta);(n_1, n_2+\delta), n_3}_{(p_1, p_2+2\delta), p_3;(q_1, q_2+2\delta), q_3}\left[\begin{array}{c|l} x & (a_{p_1});\Delta(\delta, 1-\zeta-u), (c_{p_2}), \Delta(\delta, 1-\zeta+u) \\ & (e_{p_3}) \\ y & (b_{q_1});\Delta(\delta, 1/2-\zeta), (d_{q_2}), \Delta(\delta, 1-\zeta) \\ & (f_{q_3}) \end{array}\right] -$$

$$- G^{(m_1, m_2+\delta);(n_1, n_2+\delta), n_3}_{(p_1, p_2+2\delta), p_3;(q_1, q_2+2\delta), q_3}\left[\begin{array}{c|l} x & (a_{p_1}), \Delta(\delta, -\zeta-u), (c_{p_2}), \Delta(\delta, 2-\zeta+u) \\ & (e_{p_3}) \\ y & (b_{q_1});\Delta(\delta, 1/2-\zeta), (d_{q_2}), \Delta(\delta, 1-\zeta) \\ & (f_{q_3}) \end{array}\right].$$

(*iii*) When $u=0$, we get from (2. 3) and (2. 4) respectively,

$$\text{(3. 5)} \qquad \int_0^{\pi} (\sin\theta/2)^{-2\zeta}\, G^{(m_1, m_2);(n_1, n_2), n_3}_{(p_1, p_2), p_3;(q_1, q_2), q_3}\left[\begin{array}{c|l} x\sin\theta/2)^{-2\delta} & (a_{p_1});(c_{p_2}) \\ & (e_{p_3}) \\ y & (b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array}\right] d\theta =$$

$$= \sqrt{\frac{\pi}{\delta}}\, G^{(m_1+\delta, m_2);(n_1, n_2), n_3}_{(p_1+\delta, p_2), p_3;(q_1+\delta, q_2), q_3}\left[\begin{array}{c|l} x & (a_{p_1}), \Delta(\delta, 1-\zeta);(c_{p_2}) \\ & (e_{p_3}) \\ y & \Delta(\delta, 1/2-\zeta), (b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array}\right].$$

The conditions of validity are same as for (2. 3)

$$\text{(3. 6)} \qquad \int_0^{\pi} (\sin\theta/2)^{-2\zeta}\, G^{(m_1, m_2);(n_1, n_2), n_3}_{(p_1, p_2), p_3;(q_1, q_2), q_3}\left[\begin{array}{c|l} x & (a_{p_1});(c_{p_2}) \\ & (e_{p_3}) \\ y(\sin\theta/2)^{-2\delta} & (b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array}\right] d\theta$$

$$= \sqrt{\frac{\pi}{\delta}}\, G^{(m_1, m_2+\delta);(n_1, n_2), n_3}_{(p_1, p_2+\delta), p_3;(q_1, q_2+\delta), q_3}\left[\begin{array}{c|l} x & (a_{p_1});(c_{p_2}), \Delta(\delta, 1-\zeta) \\ & (e_{p_3}) \\ y & (b_{q_1});\Delta(\delta, 1/2-\zeta), (d_{q_2}) \\ & (f_{q_3}) \end{array}\right].$$

The conditions of validity are same as for (2. 4).

(*iv*) Putting $m_2 = q_2 = 1$, $n_2 = n_3 = p_2 = p_3 = q_3 = 0$ and making use of the formula [3, 1. 5], viz

$$\text{(3. 7)} \qquad G^{(m, 1);(n, 0), 0}_{(p, 0), 0;(q, 1), 0}\left[\begin{array}{c|l} x & (a_p);\, - \\ & - \\ z & (b_q);\ 0 \\ & - \end{array}\right] = e^{-y}\, G^{m, n}_{p, q}\left[x\left|\begin{array}{l}(a_p)\\(b_q)\end{array}\right.\right]$$

we get from (2. 1),

$$\int_0^{\pi} \sin(2u+1)\theta\,(\sin\theta)^{1-2\zeta}\, G^{m_1,n_1}_{p_1,q_1}\left[x(\sin\theta)^{-2\delta}\,\middle|\,\begin{matrix}(a_{p_1})\\(b_{q_1})\end{matrix}\right] d\theta \tag{3. 8}$$
$$= \sqrt{\frac{\pi}{\delta}}\, G^{m_1+\delta,n_1+\delta}_{p_1+2\delta,q_1+2\delta}\left[x\,\middle|\,\begin{matrix}\Delta(\delta,1-\zeta-u),(a_{p_1}),\ \Delta(\delta,2-\zeta+u)\\ \Delta(\delta,3/2-\zeta),(b_{q_1}),\ \Delta(\delta,1-\zeta)\end{matrix}\right]$$

where

$$2(m_1+n_1)>p_1+q_1,\qquad \mathrm{Re}[3-2\zeta+2\delta(1-a_j)]>0;\qquad j=1,2,\cdots,n_1$$
$$|\arg x|<\left[m_1+n_1-\frac{1}{2}(p_1+q_1)\right];\ u=0,1,2,\cdots$$

(3. 8) is same as [2, p. 705, (2. 5)]. Further putting $\delta=1$ and $\zeta=0$, (3. 8) reduces to a known result [5, p. 151].

Similarly specialising the parameters, we get from (2. 3) another known result [5, p. 151].

4. Fourier Series.

The FOURIER series to be established are

$$(\sin\theta)^{1-2\zeta}\, G^{(m_1,m_2);(n_1,n_2),n_3}_{(p_1,p_2),p_3;(q_1,q_2),q_3}\left[\begin{array}{c|l} x(\sin\theta)^{-2\delta} & (a_{p_1});(c_{p_2})\\ & (e_{p_3})\\ y & (b_{q_1});(d_{q_2})\\ & (f_{q_3})\end{array}\right] \tag{4. 1}$$
$$= \frac{2}{\sqrt{\pi\delta}}\sum_{r=0}^{\infty} G^{(m_1+\delta,m_2);(n_1+\delta,n_2),n_3}_{(p_1+2\delta,p_2),p_3;(q_1+2\delta,q_2),q_3}\left[\begin{array}{c|l} x & \Delta(\delta,1-\zeta-r),(a_{p_1}),\Delta(\delta,2-\zeta+r);(c_{p_2})\\ & (e_{p_3})\\ y & \Delta(\delta,3/2-\zeta),(b_{q_1}),\Delta(\delta,1-\zeta);(d_{q_2})\\ & (f_{q_3})\end{array}\right]\sin(2r+1)\theta\,.$$

The conditions of validity are same as for (2. 1)

$$(\sin\theta)^{1-2\zeta}\, G^{(m_1,m_2);(n_1,n_2),n_3}_{(p_1,p_2),p_3;(q_1,q_2),q_3}\left[\begin{array}{c|l} x & (a_{p_1});(c_{p_2})\\ & (e_{p_3})\\ y(\sin\theta)^{-2\delta} & (b_{q_1});(d_{q_2})\\ & (f_{q_3})\end{array}\right] \tag{4. 2}$$
$$= \frac{2}{\sqrt{\pi\delta}}\sum_{r=0}^{\infty} G^{(m_1,m_2+\delta);(n_1,n_2+\delta),n_3}_{(p_1,p_2+2\delta),p_3;(q_1,q_2+2\delta),q_3}\left[\begin{array}{c|l} x & (a_{p_1});\Delta(\delta,1-\zeta-r),(c_{p_2}),\Delta(\delta,2-\zeta+r)\\ & (e_{p_3})\\ y & (b_{q_1});\Delta(\delta,3/2-\zeta),(d_{q_2}),\Delta(\delta,1-\zeta)\\ & (f_{q_3})\end{array}\right]\sin(2r+1)\theta.$$

The conditions of validity are same as for (2. 2)

$$(4.\,3)\qquad (\sin\theta/2)^{-2\zeta}\, G^{(m_1,m_2);(n_1,n_2),n_3}_{(p_1,p_2),p_3;(q_1,q_2),q_3}\left[\begin{array}{c|l} x(\sin\theta/2)^{-2\delta} & (a_{p_1});(c_{p_2}) \\ & (e_{p_3}) \\ y & (b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array}\right]=$$

$$=\frac{1}{\sqrt{\pi\delta}}\, G^{(m_1+\delta,m_2);(n_1,n_2),n_3}_{(p_1+\delta,p_2),p_3;(q_1+\delta,q_2),q_3}\left[\begin{array}{c|l} x & (a_{p_1}),\Delta(\delta,1-\zeta);(c_{p_2}) \\ & (e_{p_3}) \\ y & \Delta(\delta,1/2-\zeta),(b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array}\right]+$$

$$+\sum_{r=1}^{\infty}\frac{2}{\sqrt{\pi\delta}}\, G^{(m_1+\delta,m_2);(n_1+\delta,n_2),n_3}_{(p_1+2\delta,p_2),p_3;(q_1+2\delta,q_2),q_3}\left[\begin{array}{c|l} x & \Delta(\delta,1-\zeta-r),(a_{p_1}),\Delta(\delta,1-\zeta+r);(c_{p_2}) \\ & (e_{p_3}) \\ y & \Delta(\delta,1/2-\zeta),(b_{q_1}),\Delta(\delta,1-\zeta);(d_{q_2}) \\ & (f_{q_3}) \end{array}\right]\cos r\theta\,.$$

The conditions of validity are same as for (2. 3)

$$(\sin\theta/2)^{-2\zeta}\, G^{(m_1,m_2);(n_1,n_2),n_3}_{(p_1,p_2),p_3;(q_1,q_2),q_3}\left[\begin{array}{c|l} x & (a_{p_1});(c_{p_2}) \\ & (e_{p_3}) \\ y(\sin\theta/2)^{-2\delta} & (b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array}\right]$$

(4. 4)

$$=\frac{1}{\sqrt{\pi\delta}}\, G^{(m_1,m_2+\delta);(n_1,n_2),n_3}_{(p_1,p_2+\delta),p_3;(q_1,q_2+\delta),q_3}\left[\begin{array}{c|l} x & (a_{p_1});(c_{p_2}),\Delta(\delta,1-\zeta) \\ & (f_{q_3}) \\ y & (b_{q_1});\Delta(\delta,1/2-\zeta),(d_{q_2}) \\ & (f_{q_3}) \end{array}\right]+$$

$$+\sum_{r=1}^{\infty}\frac{2}{\sqrt{\pi\delta}}\, G^{(m_1,m_2+\delta);(n_1,n_2+\delta),n_3}_{(p_1,p_2+2\delta),p_3;(q_1,q_2+2\delta),q_3}\left[\begin{array}{c|l} x & (a_{p_1});\Delta(\delta,1-\zeta-r),(c_{p_2}),\Delta(\delta,1-\zeta+r) \\ & (e_{p_3}) \\ y & (b_{q_1});\Delta(\delta,1/2-\zeta),(d_{q_2}),\Delta(\delta,1-\zeta) \\ & (f_{q_3}) \end{array}\right]\cos(r\theta).$$

The conditions of validity are same as for (2. 4).

PROOF. To prove (4. 1), let

$$(4.\,5)\qquad f(\theta)=(\sin\theta)^{1-2\zeta}\, G^{(m_1,m_2);(n_1,n_2)n_3}_{(p_1,p_2);p_3;(q_1,q_2),q_3}\left[\begin{array}{c|l} x(\sin\theta)^{-2\delta} & (a_{p_1});(c_{p_2}) \\ & (e_{p_3}) \\ y & (b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array}\right]=\sum_{r=0}^{\infty}C_r\sin(2r+1)\theta\,.$$

Equation (4. 5) is valid since $f(\theta)$ is continuous and of bounded variation in the interval $(0, \pi)$ when $\mathrm{Re}(1-2\zeta) \geqq 0$.

Multiplying both the sides of (4. 5) by $\sin(2u+1)\theta$ and integrating with respect to θ from 0 to π, we get

$$\int_0^{\pi} (\sin\theta)^{1-2\zeta} \sin(2u+1)\theta \, G^{(m_1, m_2);(n_1, n_2), n_3}_{(p_1, p_2), p_3;(q_1, q_2), q_3} \left[\begin{array}{c|l} x(\sin\theta)^{-2\delta} & (a_{p_1});(c_{p_2}) \\ & (e_{p_3}) \\ y & (b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array} \right] d\theta$$

$$= \sum_{r=0}^{\infty} C_r \int_0^{\pi} \sin(2u+1)\theta \sin(2r+1)\theta \, d\theta .$$

Now using (2. 1) and the orthogonality property of the sine functions, we get

$$(4.\ 6) \quad C_r = \frac{2}{\sqrt{\pi\,\delta}} G^{(m_1+\delta, m_2);(n_1+\delta, n_2), n_3}_{(p_1+2\delta, p_2), p_3;(q_1+2\delta, q_2), q_3} \left[\begin{array}{c|l} x & \Delta(\delta, 1-\zeta-r), (a_{p_1}), \Delta(\delta, 2-\zeta+r);(c_{p_2}) \\ & (e_{p_3}) \\ y & \Delta(\delta, 3/2-\zeta), (b_{q_1}), \Delta(\delta, 1-\zeta);(d_{q_2}) \\ & (f_{q_3}) \end{array} \right].$$

From (4. 5) and (4. 6), the formula (4. 1) is proved.

(4. 2) is also proved by the same method as above and using (2. 2).

To prove (4. 3), let

$$(4.\ 7) \quad f(\theta) = (\sin\theta/2)^{-2\zeta} G^{(m_1, m_2);(n_1, n_2), n_3}_{(p_1, p_2), p_3;(q_1, q_2), q_3} \left[\begin{array}{c|l} x(\sin\theta/2)^{-2\delta} & (a_{p_1});(c_{p_2}) \\ & (e_{p_3}) \\ y & (b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array} \right] = C_0 + \sum_{r=1}^{\infty} C_r \cos(r\theta).$$

Integrating (4. 7) with respect to θ from 0 to π and using (3. 5), we get

$$(4.\ 8) \quad C_0 = \frac{1}{\sqrt{\pi\,\delta}} G^{(m_1, \delta, m_2);(n_1, n_2), n_3}_{(p_1+\delta, p_2), p_3;(q_1+\delta, q_2), q_3} \left[\begin{array}{c|l} x & (a_{p_1}), \Delta(\delta, 1-\zeta);(c_{p_2}) \\ & (e_{p_3}) \\ y & \Delta(\delta, 1/2-\zeta), (b_{q_1});(d_{q_2}) \\ & (f_{q_3}) \end{array} \right].$$

$$(4.\ 9) \quad C_r = \frac{2}{\sqrt{\pi\,\delta}} G^{(m_1+\delta, m_2);(n_1+\delta, n_2), n_3}_{(p_1+2\delta, p_2), p_3;(q_1+2\delta, q_2), q_3} \left[\begin{array}{c|l} x & \Delta(\delta, 1-\zeta-r), (a_{p_1}), \Delta(\delta, 1-\zeta+r);(c_{p_2}) \\ & (e_{p_3}) \\ y & \Delta(\delta, 1/2-\zeta), (b_{q_1}), \Delta(\delta, 1-\zeta);(d_{q_2}) \\ & (f_{q_3}) \end{array} \right]$$

From (4. 7), (4. 8) and (4. 9), the formula (4. 3) is proved. Formula (4. 4) is also proved by the same process and using (2. 4).

Particular cases. Specialising the parameters and making use of (3. 7), we get the known results [2, p. 707, (3. 7)] and [5, p. 149 (1. 1)] as particular cases of (4. 1). Similarly the result [5, p. 149 (1. 2)] can be obtained as particular case of (4. 3).

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